Algorithm Design for A Class of Base Station Location Problems in Sensor Networks

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Abstract. Base station placement has significant impact on sensor network performance. Despite its significance, results on this problem remain limited, particularly theoretical results that can provide performance guarantee. This paper proposes a set of procedure to design $(1-\varepsilon)$ approximation algorithms for base station placement problems under any desired small error bound $\varepsilon > 0$. It offers a general framework to transform infinite search space to a finite-element search space with performance guarantee. We apply this procedure to solve two practical problems. In the first problem where the objective is to maximize network lifetime, an approximation algorithm designed through this procedure offers $1/\varepsilon^2$ complexity reduction when compared to a state-of-the-art algorithm. This represents the best known result to this problem. In the second problem, we apply the design procedure to address base station placement problem when the optimization objective is to maximize network capacity. Our $(1-\varepsilon)$ approximation algorithm is the first theoretical result on this problem.

Keywords: Base station placement, network lifetime, network capacity, wireless sensor networks, approximation algorithm, complexity

1. Introduction

An important characteristic for wireless sensor networks is that many performance measures (e.g., lifetime, capacity) are highly dependent upon the topology of the actual physical network. For instance, the energy expenditure to transmit data from one node to another node not only depends on the data bit rate, but also on the physical distance between the two nodes. Consequently, it is important to understand the impact of location related issues on network performance and take possible steps to optimize performance starting from network deployment stage.

This paper focuses on the important problem of base station placement such that certain network performance objectives can be optimized. Although there is active research on maximizing network lifetime (see, e.g., [1, 3, 9, 19]) or network capacity (see, e.g., [6, 10, 13, 15, 20]), most of these efforts consider a sensor network under a *given* physical topology. Indeed, the location problems for base stations have been very difficult to analyze (shown to be NP-complete in [2]) and only very special cases have been investigated for optimal placement, e.g., single-hop communication between sensor node and base station [11] or special grid topology [2].

In a very recent and important work [5], Efrat et al. developed the first $(1-\varepsilon)$ approximation algorithm for base station placement (with the objective of maximizing network lifetime). Unfortunately, the computational complexity associated with this algorithm is quite high. Further, the proposed approximation solution procedure in [5] is specific to the network lifetime problem, which cannot be easily extended to address base station placement problems for other network performance objectives.

Our efforts in this paper are inspired by the work in [5]. In this paper, we aim to achieve the following two objectives. First, for the base station placement problem with network lifetime objective studied in [5], we aim to design an approximation algorithm with significant reduction on computation complexity. Second, we aim to develop a design procedure for $(1-\varepsilon)$ approximation algorithms that can be applied to solve a broad class of optimization problems. To keep our scope within base station placement problems for sensor networks, we will show how such a procedure can be used to design $(1-\varepsilon)$ approximation algorithms with a different optimization objective, e.g., network capacity.

The proposed design procedure in this paper meets both of the above two objectives. Our contribution in this paper is theoretical in nature and represents fundamental results in sensor networks. The proposed design procedure consists of four phases, once successfully applied to a specific optimization problem, can provide an $(1-\varepsilon)$ approximation algorithm to some of the most difficult optimization problems (NP-complete). A basic idea in this procedure is to replace an infinite search space for each variable by a finite-element search space but with a guaranteed bound on possible loss in performance. To prevent the search space (for all variables) from increasing exponentially with the number of variables (as in [5]), an important contribution in our design procedure is a *complexity reduction technique,* which exploits the potential overlap among the elements in the search space. Specifically, we explore the product relationship among the variables and design the search space for each of them in the form of a *geometric progression*.[1] By identifying a common factor among these geometric progressions, we show it is possible to reduce the total number of elements in the search space significantly.

As applications of the design procedure, we develop approximation algorithms for two different base station

[1] A geometric progression is a sequence, such as $1, 3, 9, 27, 81$, in which any term is its previous term multiplied by a common factor.

placement problems. The first problem is the same as the one in [5], i.e., how to place the base stations so that network lifetime can be maximized. Specifically, we show how to design an approximation algorithm for base station placement such that network lifetime is at least $(1-\varepsilon)$ of the maximum network lifetime, for any desired small approximation bound $\varepsilon > 0$. The computational complexity of our new approximation algorithm is $1/\varepsilon^2$ lower than the algorithm proposed in [5]. This represents the best known result on this problem.

To demonstrate the utility of the design procedure, we show how it can be used to design approximation algorithms for other difficult optimization problems. In the second problem, we consider how to place the base stations such that the weighted network capacity can be maximized, under the condition that each node must meet a common lifetime requirement. Although this problem also considers base station placement, it has different objective function and thus requires different formulation and solution. We show that the proposed design procedure can also be successfully applied, although the details are problem-specific. Again, we design an approximation algorithm for this problem such that the weighted network capacity is at least $(1-\varepsilon)$ times the maximum. This represents the first theoretical result for this problem.

The rest of the paper is organized as follows. Section 2 presents the sensor network model used in this study and describes two base station placement problems. In Section 3, we lay a theoretical foundation for the design of $(1-\varepsilon)$ approximation algorithms. In Section 4, we apply the design procedure to solve base station placement problem with the objective of maximizing network lifetime, while in Section 5, we apply the same design procedure to solve base station placement problem when the objective is to maximize network capacity. Section 6 reviews related work and Section 7 concludes this paper.

2. Network Model and Base Station Placement Problems

We consider a sensor network consisting of N sensor nodes deployed over a two-dimensional area. The location of each sensor node is fixed and the initial energy on sensor node i is denoted as e_i. We assume there are M base stations that need to be deployed in the area to collect sensing data. The case where $M = 1$ represents a single base station, is perhaps most common. But our algorithms developed in this paper can also handle the general case when $M > 1$, i.e., multiple base stations.

In this paper, we focus on the energy consumption due to communications (i.e., data transmission and reception). Suppose sensor node i transmits data to sensor node j with a rate of f_{ij} b/s. Then we model the transmission power at sensor node i as [8]:

$$p^t_{ij} = c_{ij} \cdot f_{ij} \ . \tag{1}$$

c_{ij} is the cost on link (i, j), and can be modeled as

$$c_{ij} = \alpha + \beta \cdot d^n_{ij} \ , \tag{2}$$

where α and β are two constant terms, d_{ij} is the physical distance between sensor nodes i and j, n is the path loss index.

The power consumption at the receiving sensor node i can be modeled as [8]:

$$p^r_i = \rho \cdot \sum_{k \neq i} f_{ki} \ , \tag{3}$$

where f_{ki} (also in b/s) is the incoming bit-rate received by sensor i from sensor k. It is easy to observe from (1), (2), and (3) that the locations for the base stations as well as data routing in the network have a profound impact on energy consumption behavior among the nodes.

The above transmission and reception energy model assumes a contention-free MAC protocol, where interference from simultaneous transmission can be effectively minimized or avoided. For deterministic rate traffic pattern model in this paper, a contention-free MAC protocol is fairly easy to design (see, e.g., [14]) and its discussion is beyond the scope of this paper.

The focus of this paper is to investigate base station placement problems in sensor networks. Clearly, how the base station should be placed depends on the particular network performance objective that we wish to optimize. In this paper, we consider the network lifetime and capacity objectives, each of which has attracted great interest.

- In the first problem, each sensor node i produces data rate r_i that needs to be routed to the base stations. The problem is how to place the base stations and arrange data routing such that the network lifetime is maximized, where network lifetime is defined as the time until any sensor node uses up its energy.
- In the second problem, the network lifetime requirement is T and data rate r_i at each sensor node i is an optimization variable. The problem is how to locate the base stations and arrange data routing such that the weighted network capacity, $\sum_{i=1}^{N} w_i r_i$, is maximized, where w_i is a pre-specified weight for sensor node i.

In addition to the above two problems, we conjecture the design procedure outline in the next section can also be applied to solve other hard optimization problems involving infinite search space. Table I lists all notation used in this paper.

3. A Procedure for the Design of (1-ε) Approximation Algorithms Based on Complexity Reduction Technique

The base station placement problems discussed in the last section involve optimizing objectives that are dependent

Table I. Notation

General Notation	
N	Number of sensor nodes in the network
M	Number of base stations
B_m	Denotes the m-th base station
e_i	Initial energy at sensor i
d_{ij} (or d_{i,B_m})	Distance between sensor i and sensor j (or base station B_m)
ρ	Power consumption coefficient for receiving data
c_{ij} (or c_{i,B_m})	Power consumption coefficient for transmitting data from sensor i to sensor j (or base station B_m)
α, β	Constant terms in transmission power consumption
n	Path loss index
T	Network lifetime
T_S	The maximum network lifetime when base stations can only be placed at the location of a sensor node
t_i	Sensor i's longevity
W	Weighted capacity
w_i	The weight assigned to sensor i
r_i	Sensing data rate produced at sensor i
$r_{\min}$ (or $r_{\max}$)	The minimum (or maximum) sensing data rate produced at a sensor node
f_{ij} (or f_{i,B_m})	Data rate from sensor i to sensor j (or base station B_m)
f^l	Data rate for flow l
V_{ij} (or V_{i,B_m})	Total data volume from sensor i to sensor j (or base station B_m)
e^t_{ij} (or e^t_{i,B_m})	Energy used for transmission by sensor i on link (i,j) (or link (i,B_m))
e^r_{ij}	Energy used for reception by sensor j on link (i,j)
$(e^l_{ij})^t$ (or $(e^l_{i,B_m})^t$)	Energy used for transmission on sensor i for flow l on link (i,j) (or link (i,B_m))
$(e^l_{ij})^r$	Energy used for reception by sensor j for flow l on link (i,j)
θ_{i,B_m}	Phase of base station B_m when sensor i is referred as origin
ε	Desired small approximation error from the optimum, $\varepsilon > 0$
Notation Specific for the Complexity Reduction Technique	
$f(x)$	Objective function in the optimization problem
Γ	A finite-element search space for x
L	Total number of y_k variables
$g(y_1, y_2, \cdots, y_L)$	A function to express x in terms of y_k's
Λ_k	A finite-element search space for y_k
ε_k	Desired small approximation error due to Λ_k
$\hat{g}(z)$	A function used to compute x from z
Ω	A finite-element search space for z

on several factors. We can view the dependency relationship between the objective and those factors as a function, which, due to its complexity, may not be explicitly expressed in a closed form. In this section, we outline a design procedure for a class of approximation algorithms that are particularly useful to solve such hard optimization problems. For the ease of discussion, we only discuss how to maximize a function $f(x)$ with one variable x in this section. The case where x is a vector can be easily generalized following the same procedure.

In Section 3.1, we outline a design procedure for $(1-\varepsilon)$ approximation algorithms by limiting the search space of x into a set Γ consisting of finite elements while the maximum objective value $f(x)$ among all $x \in \Gamma$ is at least $(1-\varepsilon)$ of the maximum. Since it is usually very difficult to construct this finite-element set Γ directly, we resort to an effective approach via divide-and-conquer. Specifically, we express x in terms of some variables y_k, i.e., $x = g(y_1, y_2, \cdots, y_L)$, where we can construct the search space of each y_k as a finite-element set Λ_k under a so-called ε_k*-mapping* criterion defined in Section 3.1. By setting $\sum_{k=1}^{L} \varepsilon_k = \varepsilon$, we show that Γ (for x) can be obtained from these Λ_k (for y_k's) and the maximum objective value $f(x)$ among all $x \in \Gamma$ is at least $(1-\varepsilon)$ of the maximum.

The procedure in Section 3.1 may have high computational complexity (the number of elements in the search space increases exponentially with L). In Section 3.2, we aim to reduce its computational complexity. In particular, we propose a complexity reduction technique in algorithm design, which explores the relationship among the finite-element set for each y_k. Specifically, we construct each Λ_k as a *geometric progression* with factor q_k, while choosing ε_k's to satisfy $q_1 = q_2 = \cdots = q_L = q$ and $\sum_{k=1}^{L} \varepsilon_k = \varepsilon$. We show that doing so can significantly reduce the computational complexity (the number of elements in the search space is linear with L).

3.1. Design Procedure: Basic Idea

We now present the basic idea in the design procedure for $(1-\varepsilon)$ approximation algorithms. For variable x, the search space to find the maximum $f(x)$ is a set with infinite elements. Since it is impossible to check all elements in an infinite-element set, we aim to limit the search space to a finite-element set, say Γ. As doing so may compromise the optimality of the solution, *the key is to show that the finite-element set contains at least one element that is at least* $(1-\varepsilon)$ *of the maximum.* Note that there is a trade-off between performance (ε) and complexity ($|\Gamma|$), where $|\Gamma|$ is the number of elements in set Γ. The better performance (the smaller ε) we want, the higher complexity (the larger the search space $|\Gamma|$) the algorithm has. The basic idea in this design procedure is the following.

1. Set up a mathematic model for the optimization problem, i.e., maximize $f(x)$, where $f(x)$ can be computed in polynomial-time for any given x.
2. For a given $\varepsilon > 0$, construct a finite-element set Γ that meets the following criterion: for any given x, there exists a $\hat{x} \in \Gamma$ such that $f(\hat{x}) \geq (1-\varepsilon)f(x)$. We call this *$\varepsilon$-mapping* criterion.
3. By examining all the elements in the finite-element set Γ, we choose x^*_Γ that has the maximum objective $f(x^*_\Gamma)$ as the final $(1-\varepsilon)$ approximation solution.

Whether or not it is possible to construct such a set is problem specific and is the main challenge in the design of $(1-\varepsilon)$ approximation algorithms. Suppose we can do this for a specific problem, then the following result holds.

Lemma *1.* If Γ meets the ε-mapping criterion, then x^*_Γ is a $(1-\varepsilon)$ approximation solution, i.e., $f(x^*_\Gamma) \geq (1-\varepsilon)f(x^*)$.

Proof. Since Γ meets the ε-mapping criterion, then for the special case of $x = x^*$, where x^* is the optimal solution, we know that there exists a $\hat{x}^* \in \Gamma$ such that $f(\hat{x}^*) \geq (1-\varepsilon)f(x^*)$. Since $f(x^*_\Gamma) \geq f(\hat{x}^*) \geq (1-\varepsilon)f(x^*)$, x^*_Γ is a $(1-\varepsilon)$ approximation solution.

As discussed, $f(\cdot)$ can be a very complex function and even may not be explicit (as in the two problems that we will solve in Sections 4 and 5). As a result, a direct construction of a finite-element set Γ that meets the ε-mapping criterion may be extremely difficult, if at all possible. Under such circumstance, it is necessary to explore other approach.

The approach that we use is *divide-and-conquer,* which breaks up a hard problem into a number of easier subproblems. Specifically, although we could not construct a finite-element set Γ for x that meets the ε-mapping criterion, it may be possible to express x as a function of some other variables, i.e., $x = g(y_1, y_2, \cdots y_L)$, such that it is possible to construct finite-element set Λ_k for each y_k, $k = 1,\ 2\ \cdots,\ L$, that meets ε_k-mapping criterion, which is defined as follows.

Definition *1.* **(ε_k-Mapping Criterion)** A finite-element set Λ_k for y_k, $1 \leq k \leq L$, is said to meet the ε_k-mapping criterion if for any given $x = g(y_1, y_2, \cdots, y_k, \cdots, y_L)$, there exists a $\hat{x} = g(\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_k, \cdots \hat{y}_L)$ with $\hat{y}_j = y_j$ for $1 \leq j \leq k-1$, $\hat{y}_k \in \Lambda_k$, and $f(\hat{x}) \geq (1-\varepsilon_k)f(x)$.

Note that in ε_k-mapping, we restrict the first $k-1$ variables to be identical to those under x. As we will show, this requirement is crucial to ensure that Lemma 2 will hold.

As a result, we can define a finite-element set Γ based on these sets Λ_k and show that it meets the ε-mapping criterion. In other words, Step 2 in the above design procedure can be further divided into the following two sub-steps.

- Express x as $x = g(y_1, y_2, \cdots, y_L)$ such that (i) $g(y_1, y_2, \cdots, y_L)$ can be computed in polynomial time; and (ii) for any given $\varepsilon_k > 0$, $1 \leq k \leq L$, we can construct a finite-element set Λ_k for y_k that meets the ε_k-mapping criterion.
- For the given $\varepsilon > 0$, determine the values for ε_k such that $\sum_{k=1}^{L} \varepsilon_k = \varepsilon$. Let $\Gamma = \{g(y_1, y_2, \cdots, y_L) : y_k \in \Lambda_k, 1 \leq k \leq L\}$.

The main task in the above design procedure is thus to construct Λ_k, $1 \leq k \leq L$, to meet the ε_k-mapping criterion. This construction process is problem specific, i.e., whether or not such construction is possible depends on the specific problem. In Sections 4 and 5, we show that, for the base station placement problems (with either network lifetime or network capacity objective), the construction of Λ_k is possible.

Now suppose that we have successfully constructed Λ_k for all $1 \leq k \leq L$, each meeting its ε_k-mapping criterion, then the following lemma is true.

Lemma *2.* Γ is a finite-element set with $|\Gamma| = O(\prod_{k=1}^{L} |\Lambda_k|)$ and meets the ε-mapping criterion, i.e., for any given solution x, there exists a solution $\hat{x} \in \Gamma$ such that $f(\hat{x})$ is at least $(1-\varepsilon)f(x)$.

Proof. $|\Gamma|$ is the number of distinct values of $g(y_1, y_2, \cdots, y_L)$ for $y_k \in \Lambda_k$, $1 \leq k \leq L$, which is at most $\prod_{k=1}^{L} |\Lambda_k|$. That is, $|\Gamma| = O(\prod_{k=1}^{L} |\Lambda_k|)$.

Instead of proving that Γ meets the ε-mapping criterion, we can prove an even stronger result by induction: for all k, $1 \leq k \leq L$, there exists a $x_k = g(y_1^{(k)}, y_2^{(k)}, \cdots, y_L^{(k)})$ such that $y_j^{(k)} \in \Lambda_j$ for $1 \leq j \leq k$ and $f(x_k) \geq (1 - \sum_{j=1}^{k} \varepsilon_j)f(x)$. Note that the result for $k = L$ is the above lemma.

We prove the above result by induction. The result for $k = 1$ is just the fact that Λ_1 meets the ε_1-mapping criterion. Now, assume that the result is true for $k = i-1$. That is, there exists a $x_{i-1} = g(y_1^{(i-1)}, y_2^{(i-1)}, \cdots, y_L^{(i-1)})$ with $y_j^{(i-1)} \in \Lambda_j$ for $1 \leq j \leq i-1$ and $f(x_{i-1}) \geq (1 - \sum_{j=1}^{i-1} \varepsilon_j)f(x)$. Based on the ε_i-mapping criterion for Λ_i, we have, for given solution x_{i-1}, there exists a $x_i = g(y_1^{(i)}, y_2^{(i)}, \cdots, y_L^{(i)})$ with $y_j^{(i)} = y_j^{(i-1)} \in \Lambda_j$ for $1 \leq j \leq i-1$, $y_i^{(i)} \in \Lambda_i$, and $f(x_i) \geq (1-\varepsilon_i)f(x_{i-1}) \geq (1-\varepsilon_i)\cdot(1-\sum_{j=1}^{i-1}\varepsilon_j)f(x) > (1-\sum_{j=1}^{i}\varepsilon_j)f(x)$. Thus, the result is also true for $k = i$. This completes the proof.

3.2. Complexity Reduction Technique and Complete Design Procedure

There is one problem associated with the approximation algorithm developed in the last section. Although the solution is a $(1-\varepsilon)$ approximation solution, the complexity increases exponentially with L. Even though L is a small number, $|\Gamma|$ may still be a very large number. In this section, we aim to reduce such complexity.

Note that the search space (Γ) derived in the last section follows a "brute force" product of Λ_k's. This is because the construction of Λ_k's are done independently. As a result, any set of $y_1, y_2, \cdots, y_L$ may produce a distinct value of $x = g(y_1, y_2, \cdots, y_L)$ and we will have to check a set Γ of size $O(\prod_{k=1}^{L} |\Lambda_k|)$.

The main idea in our complexity reduction technique is as follows. If we could construct all the Λ_k's intelligently by synthesizing some common factor among the y_k's, then we could reduce the size of the search space. Specifically, we exploit the relationship between x and certain polynomial product of all y_k's, $1 \le k \le L$, and design each Λ_k as a *geometric progression* such that all these geometric progressions for Λ_k's share a common factor. That is, we construct the finite-element set Λ_k for y_k into the following geometric progression form: $\{a_k q_k^{h_k} : h_k = 0, 1, \cdots, H_k\}$ (i.e., $\{a_k, a_k q_k, \cdots, a_k q_k^{H_k}\}$), where $a_k > 0$ and $q_k > 1$. It is important to choose the values for ε_k's so that not only $\sum_{k=1}^{L} \varepsilon_k = \varepsilon$ but also $q_1 = q_2 = \cdots = q_L = q$ (i.e., a common factor among all Λ_k's). As a result, the number of elements in $|\Gamma|$ can be reduced significantly, i.e., from the previous $|\Gamma| = O(\prod_{k=1}^{L} |\Lambda_k|)$ down to $O(\sum_{k=1}^{L} |\Lambda_k|)$ as we will prove shortly.

The complete steps for the design procedure can be summarized as follows.

Procedure *1.* **(Design Procedure for $(1-\varepsilon)$ Approximation Algorithm)**

- **Phase 1** Set up a mathematic model for the optimization problem, i.e., maximize $f(x)$, where $f(x)$ can be computed in polynomial-time for any given x.
- **Phase 2** Express x as $x = \hat{g}(z)$ and $z = \prod_{k=1}^{L} y_k^{p_k}$, where y_k are all non-negative variables and p_k are all constant integers, $1 \le k \le L$, such that (i) $\hat{g}(z)$ can be computed in polynomial time for any given z; and (ii) for any given $\varepsilon_k > 0$, $1 \le k \le L$, we can construct a finite-element set $\Lambda_k = \{a_k q_k^{h_k} : h_k = 0, 1, \cdots, H_k\}$ for y_k to meet the ε_k-mapping criterion, where $a_k > 0$ and $q_k > 1$.
- **Phase 3** For the given $\varepsilon > 0$, assign the values for ε_i such that (i) $q_1 = q_2 = \cdots = q_L = q$ (Note that q_k is a function of ε_k) and (ii) $\sum_{k=1}^{L} \varepsilon_k = \varepsilon$. Let $\Gamma = \{\hat{g}(z) : z \in \Omega\}$, where $\Omega = \{\prod_{k=1}^{L} y_k^{p_k} : y_k \in \Lambda_k, 1 \le k \le L\}$.
- **Phase 4** By examining all the elements in the finite-element set Γ, we choose x_Γ^* that has the maximum objective $f(x_\Gamma^*)$ as the $(1-\varepsilon)$ approximation solution.

Again, whether or not it is possible to construct Λ_k, $1 \le k \le L$, that meets the ε_k-mapping criterion is problem-specific and is the main task in applying the above design procedure. In Sections 4 and 5, we show that, for the base station placement problems (with either network lifetime or network capacity objective), the construction of Λ_k that meets the ε_k-mapping criterion is possible. Once we construct Λ_k successfully, we have the following theorem.

Theorem *1.* Γ is a finite-element set with size $|\Gamma| = O(|\Omega|) = O(\sum_{k=1}^{L} |\Lambda_k|)$ and x_Γ^* is a $(1-\varepsilon)$ approximation solution, i.e., $f(x_\Gamma^*) \ge (1-\varepsilon) f(x^*)$.

Proof. We first show $|\Gamma| = O(|\Omega|) = O(\sum_{k=1}^{L} |\Lambda_k|)$. Since $\Gamma = \{\hat{g}(z) : z \in \Omega\}$, it is clear that $|\Gamma| = O(|\Omega|)$. Note that $q_1 = q_2 = \cdots = q_L = q$, we can express y_k as $a_k q^{h_k}$, where h_k is an integer and $0 \le h_k \le H_k$. Thus, we have $z = \prod_{k=1}^{L} y_k^{p_k} = \prod_{k=1}^{L} (a_k q^{h_k})^{p_k} = \prod_{k=1}^{L} a_k^{p_k} \cdot q^{\sum_{k=1}^{L} p_k h_k}$. For different sets of $\{y_1, y_2, \cdots, y_L\}$, $\prod_{k=1}^{L} a_k^{p_k}$ is a constant term and z only depends on $\sum_{k=1}^{L} p_k h_k$. Since $0 \le \sum_{k=1}^{L} p_k h_k \le \sum_{k=1}^{L} (\max_{j=1}^{L} \{p_j\}) \cdot H_k = \max_{k=1}^{L} \{p_k\} \cdot \sum_{k=1}^{L} (|\Lambda_k| - 1)$ and $\sum_{k=1}^{L} p_k h_k$ is an integer, we have $|\Omega| \le 1 + \max_{k=1}^{L} \{p_k\} \cdot \sum_{k=1}^{L} (|\Lambda_k| - 1) = O(\sum_{k=1}^{L} |\Lambda_k|)$.

We then show that x_Γ^* is a $(1-\varepsilon)$ approximation solution. Note that the four-phase design procedure is a special case of the design procedure discussed in Section 3.1. Based on Lemma 2, we know that Γ meets the ε-mapping criterion. Then, based on Lemma 1, we know that x_Γ^* is ε-optimal, i.e., $f(x_\Gamma^*) \ge (1-\varepsilon) f(x^*)$. This completes our proof.

Remark *1. For many hard optimization problems in practice, e.g., two problems to be discussed in Sections 4 and 5, it may be impossible to identify z as a single polynomial product of* all *y_k's. In this case, among all the y_k's, we group as many y_k's as possible in the definition of z (in order to take advantage of the complexity reduction technique). For the rest of y_k's that cannot be put into the polynomial product in the definition of z, we can apply the basic idea described in Section 3.1, i.e., constructing a search space Λ_k for each of these y_k's independently to meet the ε_k-mapping criterion. As a result, $|\Gamma|$ is in the order of the product of $|\Omega|$ discussed in Theorem 1 (for those y_k's in the definition of z) and $|\Lambda_k|$'s (for those y_k's not in the definition of z). Obviously, the more y_k's that we can put into the polynomial product definition for z, the lower complexity we can achieve.*

We emphasize that a proper definition of y_k's and the construction of finite-element sets Λ_k's are challenging and, for some problems, may not be even possible. For the latter case, we declare that this design procedure is not applicable to the underlying problem. This should not come as a big disappointment, as no single design procedure can solve all hard optimization problems. But, if we are able to overcome the challenge, then the algorithm designed following this procedure is a $(1-\varepsilon)$ approximation algorithm.

4. A (1-ε) Approximation Algorithm for Maximizing Network Lifetime

We now apply the design procedure in the last section to address our first base station placement problem. The network model for this problem is given in Section 2. Recall that for this problem, we consider each sensor node i producing data rate r_i that needs to be routed to the base stations. The problem is how to place the base stations and arrange data routing such that the network lifetime is maximized, where network lifetime is defined as the time until any sensor node uses up its energy.

In Sections 4.1, 4.2, and 4.3, we follow the four phases in the design procedure to construct a $(1-\varepsilon)$ approximation algorithm. Two numerical examples are given in Section 4.4.

4.1. Phase 1

In this phase, we need to set up a mathematic model for the maximum network lifetime problem, i.e., identify x variable and $f(x)$ function. For this specific problem, x is actually a vector representing the locations of M base stations (denote x_m as the m-th component of x, $1 \le m \le M$). The objective here is the network lifetime T, which corresponds to the objective function $f(x)$. For any given x, we will show that $f(x)$ can be obtained by solving a linear programming (LP) problem (polynomial complexity).

For each sensor node $i = 1, 2, \cdots, N$, we have the following incoming/outgoing flow balance equations and energy constraints.

$$r_i + \sum_{1\le k\le N}^{k\ne i} f_{ki} = \sum_{1\le j\le N}^{j\ne i} f_{ij} + \sum_{m=1}^{M} f_{i,B_m}, \quad (4)$$

$$\rho \sum_{1\le k\le N}^{k\ne i} f_{ki}T + \sum_{1\le j\le N}^{j\ne i} c_{ij} f_{ij} T + \sum_{m=1}^{M} c_{i,B_m} f_{i,B_m} T \le e_i, \quad (5)$$

where f_{ij} (or f_{i,B_m}) denotes the bit rate from sensor node i to sensor node j (or base station B_m). The first N equations in (4) state that, at each sensor node i, the bit rate r_i (generated by sensor node i), plus the total bit rate of incoming flows from other sensors, is equal to the total bit rate of outgoing flows. The second N inequalities in (5) state that the energy required for reception and transmission at each sensor node i, at the end of network lifetime T, cannot exceed its initial energy. Our objective is to maximize T while both (4) and (5) are satisfied.

When the M base stations' locations are given, i.e., c_{i,B_m}'s are constants, we can formulate the following LP.

Maximize T
subject to

$$r_i T + \sum_{1\le k\le N}^{k\ne i} V_{ki} - \sum_{1\le j\le N}^{j\ne i} V_{ij} - \sum_{m=1}^{M} V_{i,B_m} = 0 \quad (1 \le i \le N)$$

$$\sum_{1\le k\le N}^{k\ne i} \rho V_{ki} + \sum_{1\le j\le N}^{j\ne i} c_{ij} V_{ij} + \sum_{m=1}^{M} c_{i,B_m} V_{i,B_m} \le e_i \quad (1 \le i \le N)$$

$$T, V_{ij}, V_{i,B_m} \ge 0 \quad (1 \le i, j \le N, i \ne j, 1 \le m \le M)\,.$$

where $V_{ij} = f_{ij}T$ and $V_{i,B_m} = f_{i,B_m}T$, and V_{ij} (or V_{i,B_m}) is the bit volume being sent from sensor node i to sensor node j (or base station B_m). Note that T, V_{ki}, V_{ij}, and V_{i,B_m} are variables, and that r_i, ρ, c_{ij}, c_{i,B_m}, and e_i are all constants. We now have an optimization problem in the form of an LP formulation, which can be solved in polynomial time. In other words, we have shown a mathematical model for the optimization problem, where the objective $f(x)$ (the maximum network lifetime) can be computed from any given x (the locations of the base stations) in polynomial time.

To reduce the variable space and the computational complexity of the above LP, we perform the following preprocessing before running a full-scale LP. For sensor node i, denote Q_i the set containing the nearest base-station to sensor i and all other the sensors that are within the radius from sensor i to this nearest base-station. In the case where there is a tie when more than one base stations have the same smallest distance to sensor node i, we break the tie randomly. For one-hop data transmission from node i, it is only necessary to consider nodes in Q_i as possible destination. That is, any other node outside Q_i should not be a one-hop destination since node i can otherwise send to its nearest base station (in Q_i) directly in one hop. That is, we can remove variable f_{ij} and f_{i,B_m} when $j, B_m \notin Q_i$ in the LP formulation.

The following property follows the above discussion and will be used repeatedly in the Phase 2 design of the approximation algorithm.

Property *1.* To be energy efficient, if a sensor node needs to transmit to some base stations in one hop, it is sufficient to consider the case where this sensor node transmits (in one hop) to only one base station, i.e., its nearest base station.

4.2. Phase 2

Phase 2 in the design procedure is the most challenging part. Specifically, whether or not it is possible to construct Λ_k, $1 \le k \le L$, such that each Λ_k meets the ε_k-mapping criterion, is problem specific. In this part, we fill in all the details and show that it is indeed possible for our base station placement problem.

A New Notion of Lifetime. For our problem, the network lifetime is so far defined as the time instance until any node uses up its energy. It turns out such network lifetime definition is not quite convenient in our algorithm design. Instead, we introduce a new definition, which we call "longevity" to distinguish from lifetime. Longevity definition is heavily data-centric (in contrast to lifetime, which is energy-based) and refers to either the time instance when data can no longer be forwarded over a link or a flow path. Under the longevity definition, we imagine that the energy at a node is logically partitioned into different pieces, with each piece pre-assigned (or dedicated) for either transmission to another node or receiving from a different node.

Definition *2.* **(Link Longevity)** For link (i, j), denote the transmission energy allocated for this link at node i as e_{ij}^t and the receiving energy allocated for this link at node j as e_{ij}^r. Then the link longevity is defined as $\min\left\{\frac{e_{ij}^t}{c_{ij}f_{ij}}, \frac{e_{ij}^r}{\rho f_{ij}}\right\}$.

In the above definition, for the special case when node j is a base station B_m, the receiving energy on B_m is defined as ∞. Following the link longevity definition (or more precisely, when energy at a node is allocated based on links), *node longevity* is defined as the minimum longevity among all links at this node while *network longevity* is defined as the minimum longevity among all the nodes.

Definition *3.* **(Flow Longevity)** Define f^l the bit rate for a flow originating from a sensor node to a base station by traversing a path l. For each link (i, j) that is traversed by this flow, denote the transmission energy allocated to this flow at node i as $(e_{ij}^l)^t$ and the receiving energy allocated to this flow at node j as $(e_{ij}^l)^r$. The flow longevity is defined as $\min_{(i,j)\in l}\left\{\frac{(e_{ij}^l)^t}{c_{ij}f^l}, \frac{(e_{ij}^l)^r}{\rho f^l}\right\}$.

Following the flow longevity definition (or more precisely, when energy at a node is allocated based on flows), the corresponding *node longevity* can be defined as the minimum longevity among all flows originating from this node while *network longevity* is defined as the minimum longevity among all the nodes.

The following property states the relationship between the data-based network longevity definition and the (energy-based) network lifetime definition.

Property *2.* For any given solution (base station locations and data routing), the network longevity is no more than the network lifetime. Under an optimal solution, the maximum network longevity is equal to the maximum network lifetime.

It should be note that a solution under longevity definition includes not only base station locations and data routing but also energy allocation on links or flows. Under a given solution (base station locations and data routing), if the energy allocation is chosen properly, the network longevity can be equal to the network lifetime. Otherwise, the network longevity is less than the network lifetime. Based on this property, we have the following lemma.

Lemma *3.* If an algorithm is a $(1 - \varepsilon)$ approximation algorithm under network longevity criterion, then this algorithm is also a $(1 - \varepsilon)$ approximation algorithm under the network lifetime criterion.

Determination of z, $\hat{g}(z)$, and y_k. We now identify z_m, $\hat{g}_m(z_m)$, and $y_m^{(k)}$ for each x_m (the location of base station B_m). We choose z_m as a vector of the transmission cost c_{i,B_m} from each sensor node $i = 1, 2, \cdots, N$ to base station B_m. Denote z_{im} as the i-th component of z_m, we have

$$z_{im} = c_{i,B_m} .$$

For each z_{im}, we choose

$$y_{im}^{(1)} = \theta_{i,B_m} ,$$

where θ_{i,B_m} is the phase of the base station B_m (measured from the horizontal axis) when the origin is sensor node i. We now show that there is a function $\hat{g}_m(\cdot)$ such that $x_m = \hat{g}_m(z_{im}, y_{im}^{(1)})$, $1 \le i \le N$, and $\hat{g}_m(\cdot)$ can be computed in polynomial time for any given z_{im} and $y_{im}^{(1)}$. That is, the location of base station B_m (i.e., x_m) can be computed in polynomial time if we know a transmission cost c_{i,B_m} and the corresponding phase θ_{i,B_m} (i.e., z_{im} and $y_{im}^{(1)}$). Specifically, given a transmission cost c_{i,B_m}, we can calculate the distance d_{i,B_m} from sensor node i to base station B_m via (2). After we know the values of the distances d_{i,B_m}, as well as the phase θ_{i,B_m}, we can determine the location for base station B_m based on the location of sensor node i.

We now identify the rest of $y_{im}^{(k)}$ variables so that z_{im} can be expressed as a polynomial product of these $y_{im}^{(k)}$'s, $2 \le k \le L$. Denote node i's longevity as t_i. We define

$$y_{im}^{(2)} = e_{i,B_m}^t , \quad y_{im}^{(3)} = f_{i,B_m} , \quad y_{im}^{(4)} = t_i , \quad L = 4 .$$

We now show that z_{im} can be defined as

$$z_{im} = y_{im}^{(2)}(y_{im}^{(3)})^{-1}(y_{im}^{(4)})^{-1} . \tag{6}$$

Under link longevity definition, we have $t_i \le \frac{e_{i,B_m}^t}{c_{i,B_m}f_{i,B_m}}$, i.e., $c_{i,B_m} \le \frac{e_{i,B_m}^t}{f_{i,B_m}t_i}$, for each link (i, B_m). It turns out that it is sufficient to consider only the case for $c_{i,B_m} = \frac{e_{i,B_m}^t}{f_{i,B_m}t_i}$, i.e., (6). The details are explained in the next paragraph.

Note that c_{i,B_m}'s, $1 \le i \le N$, are used to determine the location for base station B_m. Assume we have $\frac{e_{i,B_m}^t}{f_{i,B_m}t_i}$ in a solution. Since $\frac{e_{i,B_m}^t}{f_{i,B_m}t_i}$ is an upper bound of each c_{i,B_m}, then the possible locations for base station B_m is the *common region* of several intersecting disks. We argue that it is sufficient to search only a boundary point for this entire region, where $c_{i,B_m} = \frac{e_{i,B_m}^t}{f_{i,B_m}t_i}$. Note that if we move base station B_m to such a point, under the same data routing and link energy allocation, the new longevity of each link (i, B_m) remains at least t_i, $1 \le i \le N$, while all other link longevities remain unchanged. Therefore, the corresponding node longevity for each node as well as the network longevity are all the same as before. We have thus obtained another solution with the same network longevity where the base station B_m is now at a boundary point of the common region. Thus, it is sufficient to search only a boundary point for solutions to maximize network longevity.

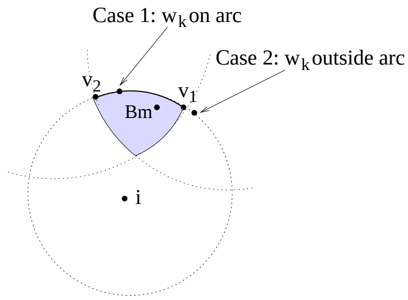


Figure 1. Constructing solution $\hat{\psi}$ by relocating base station B_m in solution ψ.

For the ease of mathematical notation, in the rest of this section, we omit the subscript im when there is no confusion. For example, we will use y_k to express $y^{(k)}_{im}$.

Construction of Λ_k. Recall that whether or not it is possible to construct Λ_k that meets ε_k-mapping criterion is problem-specific and is the main task in the design procedure described in Section 3.2. In this part, we show how to construct a finite-element set Λ_k for each y_k and show the ε_k-mapping criterion is satisfied in a series of four claims. In each claim, we construct Λ_k for y_k, $k = 1, 2, 3, 4$, such that the performance bound will decrease by no more than $1-\varepsilon_k$ when the search space for variable y_k is limited to the finite-element set Λ_k. Note that we must construct the finite-element sets Λ_2, Λ_3, and Λ_4 as geometric progressions, while Λ_1 does not have this requirement since y_1 is not in the definition of z (see Remark 1). We first construct Λ_1 for $y_1 = \theta_{i,B_m}$ as follows.

Claim *1.* **(Λ_1)** For $y_1 = \theta_{i,B_m}$ and an arbitrarily small given $\varepsilon_1 > 0$, we can construct a set $\Lambda_1 = \{h_1 a_1 : h_1 = 1, 2, \cdots, H_1\}$, with $H_1 = \lceil n\pi/\varepsilon_1 \rceil$ (where n is the path loss index) and $a_1 = 2\pi/H_1$ such that for any given solution ψ for base station placement, data routing, and energy allocation (on links) with a network longevity T, there exists a solution $\hat{\psi}$ and a sensor node i with $\theta_{i,B_m} \in \Lambda_1$ and the network longevity is $\hat{T} \geq (1-\varepsilon_1)T$.

Proof. The proof is based on construction. That is, we will move base station B_m in solution ψ and construct $\hat{\psi}$ to satisfy all requirements.

Under solution ψ, for base station B_m, we consider $\frac{e^t_{j,B_m}}{f_{j,B_m} t_j}$ for each sensor node j, $1 \leq j \leq N$. These $\frac{e^t_{j,B_m}}{f_{j,B_m} t_j}$'s define a common region by intersecting disks from different node j. As discussed, we can move B_m to any boundary point of this region while the network longevity remains unchanged. For the purpose of this proof, we only consider moving base station B_m to a point on the arc (v_1, v_2) of the region's boundary that is part of the smallest circle (i.e., circle with the smallest radius d) (see Fig. 1). Assume the center of this circle is sensor node i and denote w_k the point on this circle that is closet to B_m among these points have a phase $h_1 a_1$. We now move B_m to point w_k. There are two cases.

Case 1: We first consider the case that point w_k is on arc (v_1, v_2). As discussed, after we move B_m to this point, under the same data routing and energy allocation, the network longevity is at least T.

Case 2: We now consider the case that point w_k is outside the arc (v_1, v_2). In this case, we move base station B_m in two steps. In the first step, we move B_m to the end point v_1 of this arc toward w_k. Again, under the same data routing and energy allocation, this move will not decrease the network longevity. In the second step, we move B_m from v_1 to w_k. Since w_k is the closet point to B_m among those points that have a phase $h_1 a_1$, the phase difference of w_k and v_1 is at most $a_1/2$, which is $\pi/H_1 \leq \varepsilon_1/n$ by the definition of a_1. Thus, the length of arc (v_1, w_k) is at most $d \cdot a_1/2$. Due to this change, the distance from any sensor node j to base station B_m can increase at most $d \cdot a_1/2 \leq d_{j,B_m} \cdot \varepsilon_1/n$. Thus, the distance from any sensor node j to base station B_m can increase at most to $(d_{j,B_m} + d_{j,B_m} \cdot \varepsilon_1/n)/d_{j,B_m} = 1 + \varepsilon_1/n$. By (2), the cost c_{j,B_m} can increase at most to $\frac{\alpha+\beta[(1+\varepsilon_1/n)d_{j,B_m}]^n}{\alpha+\beta d^n_{j,B_m}} < \frac{\alpha(1+\varepsilon_1/n)^n+\beta[(1+\varepsilon_1/n)d_{j,B_m}]^n}{\alpha+\beta d^n_{j,B_m}} = (1+\varepsilon_1/n)^n$. Therefore, under the same data routing and energy allocation in solution ψ, the new longevity of link (j, B_m) from any sensor node j after this move decreases at most to $1/(1+\varepsilon_1/n)^n = [1/(1+\varepsilon_1/n)]^n > (1-\varepsilon_1/n)^n > 1-\varepsilon_1$, while all other link longevities remain unchanged. Therefore, the network longevity is $\hat{T} \geq (1-\varepsilon_1)T$.

We now construct a finite-element set Λ_2 for $y_2 = e^t_{i,B_m}$, such that the decrease in performance bound is at most ε_2 when we narrow the search space for variable y_2 into a finite-element set Λ_2.

Claim *2.* **(Λ_2)** For $y_2 = e^t_{i,B_m}$ and an arbitrarily small given $\varepsilon_2 > 0$, we can construct a set $\Lambda_2 = \{a_2 q_2^{h_2} : h_2 = 0, 1, \cdots, H_2\}$, where $a_2 = \varepsilon_2 e_i$, $q_2 = 1+\varepsilon_2$, and $H_2 = \left\lfloor \frac{\ln(1/\varepsilon_2)}{\ln(1+\varepsilon_2)} \right\rfloor$, such that for any given solution ψ for base station placement, data routing, and energy allocation (on links) with a network longevity T, there exists a solution $\hat{\psi}$ with $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$, $\hat{e}^t_{i,B_m} \in \Lambda_2$ when $\hat{e}^t_{i,B_m} > 0$, and the network longevity is $\hat{T} \geq (1-\varepsilon_2)T$.

Proof. The proof is based on construction. That is, we will revise energy allocation in solution ψ and construct $\hat{\psi}$ to satisfy all requirements. Note that we keep $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$.

For each sensor i with $e^t_{i,B_m} > 0$, we can revise energy allocation in ψ and construct $\hat{\psi}$ as follows.

$$\hat{e}^t_{i,B_m} = \begin{cases} \varepsilon_2 e_i & 0 < e^t_{i,B_m} < \varepsilon_2 e_i \\ \varepsilon_2 e_i (1+\varepsilon_2)^{h_2} & e^t_{i,B_m} \geq \varepsilon_2 e_i \end{cases} \quad (7)$$

where $h_2 = \left\lfloor \ln \frac{e^t_{i,B_m}}{\varepsilon_2 e_i} / \ln(1+\varepsilon_2) \right\rfloor$. For each link (i, j), $1 \leq i, j \leq N, i \neq j$, we revise e^t_{ij} (allocated transmission energy on node i) and e^t_{ij} (allocated receiving energy on node i)

as

$$\hat{e}^t_{ij} = (1-\varepsilon_2)e^t_{ij} \ , \quad (8)$$
$$\hat{e}^r_{ij} = (1-\varepsilon_2)e^r_{ij} \ . \quad (9)$$

Since $e^t_{i,B_m} \le e_i$, we have $h_2 \le H_2 = \left\lfloor \frac{\ln(1/\varepsilon_2)}{\ln(1+\varepsilon_2)} \right\rfloor$. That is, $\hat{e}^t_{i,B_m}$ is indeed within the set Λ_2.

We now show that this new energy allocation is feasible, i.e., the total allocated energy on each node i is no more than e_i. When $0 < e^t_{i,B_m} < \varepsilon_2 e_i$, we have

$$\sum_{m=1}^{M} \hat{e}^t_{i,B_m} \le \varepsilon_2 e_i \ , \quad (10)$$

by (7) and Property 1. Therefore,

$$\begin{aligned} &\sum_{m=1}^{M} \hat{e}^t_{i,B_m} + \sum_{1\le j\le N}^{j\ne i} \hat{e}^t_{ij} + \sum_{1\le k\le N}^{k\ne i} \hat{e}^r_{ki} \\ \le\ & \varepsilon_2 e_i + (1-\varepsilon_2)\left(\sum_{1\le j\le N}^{j\ne i} e^t_{ij} + \sum_{1\le k\le N}^{k\ne i} e^r_{ki} \right) \\ \le\ & \varepsilon_2 e_i + (1-\varepsilon_2)e_i = e_i \end{aligned}$$

The first inequality holds by (8), (9), and (10). The second inequality holds since the energy allocation in solution ψ is feasible. Thus, the energy feasibility holds when $0 < e^t_{i,B_m} < \varepsilon_2 e_i$. The proof of energy feasibility for the case when $e^t_{i,B_m} \ge \varepsilon_2 e_i$ is trivial since in the revised energy allocation, the allocated energy on each link has been decreased.

We now show that the new network longevity under the revised energy allocation is $\hat{T} \ge (1-\varepsilon_2)T$. For link longevity, there are three cases by (7), (8), and (9): (1) when $0 < e^t_{i,B_m} < \varepsilon_2 e_i$, the link longevity of link (i, B_m) increases; (2) when $e^t_{i,B_m} \ge \varepsilon_2 e_i$, the link longevity of link (i, B_m) decreases at most to $1/(1+\varepsilon_2) > 1-\varepsilon_2$; (3) for link (i, j) other than (i, B_m), its link longevity is decreased to $1-\varepsilon_2$. Taking all three cases into consideration, the new network longevity is $\hat{T} \ge (1-\varepsilon_2)T$.

We now construct a finite-element set Λ_3 for $y_3 = f_{i,B_m}$, such that the decrease in performance bound is at most ε_3 when we narrow the search space for variable y_3 into a finite-element set Λ_3.

Claim *3.* **(Λ_3)** For $y_3 = f_{i,B_m}$ and an arbitrarily small given $\varepsilon_3 > 0$, we can construct a set $\Lambda_3 = \{a_3 q_3^{h_3} : h_3 = 0, 1, \cdots, H_3\}$, with $a_3 = \frac{\varepsilon_3 r_i}{(N^2-N+2)}$, $q_3 = 1 + \frac{\varepsilon_3}{2}$, and $H_3 = \left\lceil \ln \frac{(N^2-N+2)\sum_{j=1}^{N} r_j}{\varepsilon_3 r_i} / \ln\left(1+\frac{\varepsilon_3}{2}\right) \right\rceil$, such that for any given solution ψ for base station placement and data routing with a network longevity T, there exists a solution $\hat{\psi}$ with $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$, $\hat{e}^t_{i,B_m} = e^t_{i,B_m}$, $\hat{f}_{i,B_m} \in \Lambda_3$ when $\hat{f}_{i,B_m} > 0$, and the network longevity is $\hat{T} \ge (1-\varepsilon_3)T$.

Proof. Again, the proof is based on construction. That is, we will revise the data routing in solution ψ and construct $\hat{\psi}$ to satisfy all requirements. Note that we keep $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$ and $\hat{e}^t_{i,B_m} = e^t_{i,B_m}$.

This construction has two steps. In the first step, we construct a solution $\psi^\dagger$ from ψ with $f^\dagger_{i,B_m} \ge a_3$ when $f^\dagger_{i,B_m} > 0$ and the network longevity is $T^\dagger \ge (1-\varepsilon_3/2)T$. That is, link rate from a sensor node i to base station B_m, i.e., $f^\dagger_{i,B_m}$, is no less than a_3 when it is not zero. In the second step, we construct a solution $\hat{\psi}$ from $\psi^\dagger$ with $\hat{f}_{i,B_m} \in \Lambda_3$ when $\hat{f}_{i,B_m} > 0$ and the network longevity is $\hat{T} \ge (1-\varepsilon_3/2)T^\dagger > (1-\varepsilon_3)T$.

(i) We can revise the data routing in solution ψ and construct $\psi^\dagger$. Our objective is to ensure that in solution $\psi^\dagger$, $f^\dagger_{i,B_m} \ge a_3$ when $f^\dagger_{i,B_m} > 0$. This will make necessary preparation for our efforts in (ii) to further revise the data routing in solution $\psi^\dagger$ and construct $\hat{\psi}$.

Since the revision of ψ will be done on flow level (each involving a source sensor node and a destination base station), we first decompose the data routing in ψ into flows as follows by repeatedly checking whether there is a link with the remaining rate. If the answer is yes, there must be a path from a sensor node to a base station and each link in this path has a positive remaining rate. We can identify a flow on this path with the minimum link rate among all links that belongs to this path and remove it from future flow classification in ψ. Eventually, when the remaining rates on any links become zero, we are done with flow classification on ψ. Under this flow classification, it is easy to re-allocate energy for flows on each node to achieve the same network longevity T.

We show that the number of flows is at most $(N^2 + N)/2$. Based on Property 1, there is at most one $f_{i,B_m} > 0$ for each sensor node i in ψ. Thus, there are at most $N + \frac{(N-1)N}{2} = (N^2+N)/2$ links. Since classifying one flow will remove at least one link from future classifications, the data traffic under ψ can be classified (or decomposed) into at most $(N^2 + N)/2$ flows.

We are now ready to revise this flow-based data routing in solution ψ and construct $\psi^\dagger$. The basic idea is that at each sensor node i, we shift small flows with rate $f_{i,B_m} < \frac{\varepsilon_3 r_i}{N^2-N+2}$ onto the largest flow originates from this node.

We now show that the new network longevity under the revised data routing is $T^\dagger \ge (1-\varepsilon_3/2)T$. Consider the flow with largest rate from sensor i. Since there is at least one flow from each sensor node and there are at most $(N^2 + N)/2$ flows in the network, at sensor node i, there are at most $(N^2 + N)/2 - (N - 1) = (N^2 - N + 2)/2$ flows from sensor i. Thus, the rate of the largest flow from sensor i is at least $r_i/\left(\frac{N^2-N+2}{2}\right) = \frac{2r_i}{N^2-N+2}$ in ψ. Since for each sensor node i, there is at most one $f_{i,B_m} > 0$ in solution ψ (Property 1), we move at most one flow to the largest flow that originates from sensor node i. After this move, the rate of the largest flow increases at most by a fraction of $\left(\frac{\varepsilon_3 r_i}{N^2-N+2}\right)/\left(\frac{2r_i}{N^2-N+2}\right) = \varepsilon_3/2$. Thus, the flow longevity of this largest flow decreases to at most

$1/(1+\varepsilon_3/2) > 1-\varepsilon_3/2$. Since the flow longevities of all other flows remain unchanged, solution $\psi^\dagger$ has a network longevity $T^\dagger \geq (1-\varepsilon_3/2)T$.

(ii) In the second step, we further revise the data routing in solution $\psi^\dagger$ and construct $\hat{\psi}$ as follows.

$$\hat{f}_{i,B_m} = \frac{\varepsilon_3 r_i}{N^2-N+2}\left(1+\frac{\varepsilon_3}{2}\right)^{h_3} (1\leq i\leq N, 1\leq m\leq M) \quad (11)$$
$$\hat{f}_{ij} = f^\dagger_{ij} \qquad (1\leq i,j\leq N, i\neq j)\ ,$$

where $h_3 = \left\lceil \ln \frac{(N^2-N+2)f^\dagger_{i,B_m}}{r_i\varepsilon_3} / \ln\left(1+\frac{\varepsilon_3}{2}\right)\right\rceil$. Due to the ceiling function used for h_3, $\hat{f}_{i,B_m}$ could be larger than required by flow balance. In this case, for the purpose of understanding, we can imagine that node i transmits some fictitious data (in addition to r_i) to fill up this gap. Since $f^\dagger_{i,B_m} \leq \sum_{j=1}^N r_j$, we have $h_3 \leq H_3 = \left\lceil \ln \frac{(N^2-N+2)\sum_{j=1}^N r_j}{\varepsilon_3 r_i} / \ln\left(1+\frac{\varepsilon_3}{2}\right)\right\rceil$. That is, $\hat{f}_{i,B_m}$ is indeed within the set Λ_3.

We now show that the new network longevity under the revised data routing is $\hat{T} \geq (1-\varepsilon_3/2)T^\dagger$. Since the link rate of link (i,B_m) increases at most to $1+\varepsilon_3/2$ by (11), the link longevity decreases at most to $1/(1+\varepsilon_3/2) > 1-\varepsilon_3/2$. Note that the link longevities of all other links remain unchanged, $\hat{\psi}$ has a network longevity $\hat{T} \geq (1-\varepsilon_3/2)T^\dagger \geq (1-\varepsilon_3/2)\cdot(1-\varepsilon_3/2)T > (1-\varepsilon_3)T$. This completes the proof.

We now construct a finite-element set Λ_4 for $y_4 = t_i$, such that the decrease in performance bound is no more than ε_4 when we narrow the search space for y_4 into this finite-element set Λ_4.

Claim *4.* **(Λ_4)** Denote T_S as the maximum network longevity obtained by placing base stations only at the same locations for sensor nodes. For $y_4 = t_i$ and an arbitrarily small given $\varepsilon_4 > 0$, we can construct a set $\Lambda_4 = \{a_4 q_4^{h_4} : h_4 = 0,1,\cdots,H_4\}$, with $a_4 = T_S$, $q_4 = 1+\varepsilon_4$, and $H_4 = \left\lfloor \frac{n\ln 2}{\ln(1+\varepsilon_4)}\right\rfloor$, where n is the path loss index, such that for any given solution ψ for base station placement and data routing with a network longevity T, there exists a solution $\hat{\psi}$ with $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$, $\hat{e}^t_{i,B_m} = e^t_{i,B_m}$, $\hat{f}_{i,B_m} = f_{i,B_m}$, $\hat{t}_i \in \Lambda_4$, and the network longevity is $\hat{T} \geq (1-\varepsilon_4)T$.

Proof. The proof is based on construction. That is, we will revise node longevity in solution ψ and construct $\hat{\psi}$ to satisfy all requirements. Note that we keep $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$, $\hat{e}^t_{i,B_m} = e^t_{i,B_m}$, and $\hat{f}_{i,B_m} = f_{i,B_m}$.

We first bound the search space of t_i. In Lemma 4 (see appendix), we show that $T_S \geq 2^{-n}T^*$, i.e., $T^* \leq 2^n T_S$, where T^* is the maximum network longevity under optimal solution. It is obvious that $T_S \leq T^*$. Thus, we only need to check $[T_S, 2^n T_S]$ for each t_i, since $t_i < T_S$ cannot yield an optimal solution and there is no need to make $t_i > 2^n T_S$. By decreasing the energy allocation on certain link (e.g., incoming link (k,i)), we can revise node longevity in solution ψ and construct $\hat{\psi}$ as follows.

$$\hat{t}_i = T_S(1+\varepsilon_4)^{h_4} \quad (1\leq i\leq N)\ ,$$

where $h_4 = \left\lfloor \ln \frac{t_i}{T_S} / \ln(1+\varepsilon_4)\right\rfloor$. Since we only consider the case for $t_i \leq 2^n T_S$, we have $h_4 \leq H_4 = \left\lfloor \frac{n\ln 2}{\ln(1+\varepsilon_4)}\right\rfloor$. That is, $\hat{t}_i$ is indeed within the set Λ_4.

We now show that the new network longevity under the revised node longevity is $\hat{T} \geq (1-\varepsilon_4)T$. Since node longevity of each node i decreases at most to $1/(1+\varepsilon_4) > 1-\varepsilon_4$, $\hat{\psi}$ has a network longevity $\hat{T} \geq (1-\varepsilon_4)T$. This completes the proof. □

4.3. Phases 3 and 4

We now proceed to Phase 3 and Phase 4 of the design procedure. We first determine $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 such that $\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4 = \varepsilon$ and $q_2 = q_3 = q_4 = q$. From Claims 2, 3, and 4, $q_2 = 1+\varepsilon_2$, $q_3 = 1+\varepsilon_3/2$, and $q_4 = 1+\varepsilon_4$, we choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon/5$, and $\varepsilon_3 = 2\varepsilon/5$.

For each $z = c_{i,B_m}$, we have

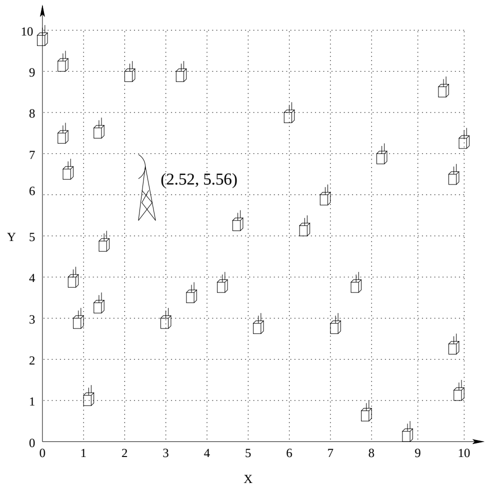
$$\begin{aligned}
\Omega &= \{y_2 y_3^{-1} y_4^{-1} : y_k \in \Lambda_k, 2\leq k\leq 4\} \\
&= \left\{\varepsilon_2 e_i (1+\varepsilon_2)^{h_2}\left[\frac{\varepsilon_3 r_i}{(N^2-N+2)}\left(1+\frac{\varepsilon_3}{2}\right)^{h_3}\right]^{-1} \left[T_S(1+\varepsilon_4)^{h_4}\right]^{-1}\right\} \\
&= \left\{\frac{\varepsilon e_i}{5}\left(1+\frac{\varepsilon}{5}\right)^{h_2}\left[\frac{2\varepsilon r_i}{5(N^2-N+2)}\left(1+\frac{\varepsilon}{5}\right)^{h_3}\right]^{-1} \left[T_S\left(1+\frac{\varepsilon}{5}\right)^{h_4}\right]^{-1}\right\} \\
&= \left\{\left(1+\frac{\varepsilon}{5}\right)^{h_2-h_3-h_4}\frac{(N^2-N+2)e_i}{2r_i T_S}\right\}\ ,
\end{aligned}$$

where $h_k = 0,1,\cdots,H_k, 2\leq k\leq 4$, and thus

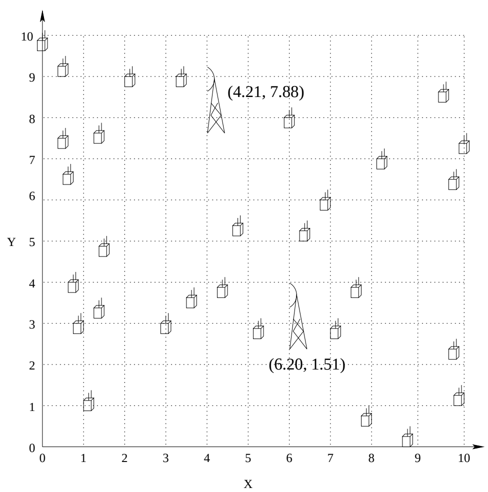
$$\begin{aligned}
|\Omega| &= O\left(\sum_{k=2}^4 |\Lambda_k|\right) = O\left(\sum_{k=2}^4 (H_k+1)\right) \\
&= O\left(\left\lceil \ln \frac{(N^2-N+2)\sum_{j=1}^N r_j}{\varepsilon_3 r_i} / \ln\left(1+\frac{\varepsilon_3}{2}\right)\right\rceil + \left\lfloor\frac{\ln(1/\varepsilon_2)}{\ln(1+\varepsilon_2)}\right\rfloor + \left\lfloor\frac{n\ln 2}{\ln(1+\varepsilon_4)}\right\rfloor + 3\right) \\
&= O\left(\left\lceil \ln \frac{5(N^2-N+2)\sum_{j=1}^N r_j}{2\varepsilon r_i} / \ln\left(1+\frac{\varepsilon}{5}\right)\right\rceil + \left\lfloor\frac{\ln(5/\varepsilon)}{\ln(1+\varepsilon/5)}\right\rfloor + \left\lfloor\frac{n\ln 2}{\ln(1+\varepsilon/5)}\right\rfloor\right) \\
&= O\left(\frac{\ln(1/\varepsilon)}{\varepsilon} + \frac{\ln(N/\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon}\right) = O\left(\frac{\ln(N/\varepsilon)}{\varepsilon}\right)\ ,
\end{aligned}$$

where we have used the fact that $\ln(1+\varepsilon/5) \approx \varepsilon/5$ for small $\varepsilon > 0$.

The set Γ for the locations of base station B_m is defined as all points with $\theta_{i,B_m} \in \Lambda_1$ and $c_{i,B_m} \in \Omega$ (or $e_{i,B_m} \in \Lambda_2$, $f_{i,B_m} \in \Lambda_3$, and $t_i \in \Lambda_4$), $1\leq i\leq N$. Based on Claims 1, 2, 3, and 4, we know that the maximum network longevity by



a. Single base station ($M = 1$)



b. Two base stations ($M = 2$)

Figure 2. Base station placement to maximize network lifetime.

checking all locations in Γ is at least $(1-\varepsilon)$ of the optimum and $|\Gamma| = O(N|\Omega||\Lambda_1|) = O(\frac{N}{\varepsilon^2}\ln\frac{N}{\varepsilon})$.

In Phase 4, a $(1-\varepsilon)$ approximation solution is obtained by examining all locations in Γ. For M base stations, the search space is $O((\frac{N}{\varepsilon^2}\ln\frac{N}{\varepsilon})^M)$.

4.4. Numerical Examples

As examples, we apply our $(1-\varepsilon)$ approximation algorithm to solve base station placement problem for $M = 1$ (single base station) and $M = 2$ (two base stations). We randomly generate a 30-node sensor network in a 10x10 area (see Fig. 2). All units are normalized in consistent to those defined in (1), (2), and (3). For the power consumption model, we set $\alpha = 1$, $\beta = 3$, $\rho = 1$, and $n = 4$. The initial energy at a node is chosen from a uniform distribution within $[50, 100]$ and the data rate is chosen from another uniform distribution within $[1, 10]$.

For a given $\varepsilon = 0.1$, the base station placements for $M = 1$ and $M = 2$ calculated by our approximation algorithm are shown in Figs. 2(a) and (b), respectively. The corresponding network lifetimes are $T = 13.50$ for $M = 1$ and $T = 30.09$ for $M = 2$.

5. A (1-ε) Approximation Algorithm for Maximizing Weighted Network Capacity

We now show that the design procedure in Section 3 can be used to address base station placement problem when the optimization objective is network capacity. In this new problem, we assume there is a weight w_i associated with each sensor node i. For a given network lifetime requirement T, we investigate how to place the base stations and perform data routing such that the weighted capacity, $\sum_{i=1}^{N} w_i r_i$, is maximized, where r_i's are variables.

Note that although the weighted capacity problem here and the network lifetime problem discussed in the last section both consider base station placement and data routing, there does not appear any duality relationship between the two problems and thus they must be solved independently. We point out that the approximation algorithm presented in this section is the first theoretical result to this problem.

In Section 4, we have given detailed exposition on how to apply the design procedure for the network lifetime problem. The development in this section builds upon the knowledge and experience in the last section and we will strive to keep our discussion as concise as possible. Readers are advised to review the last two sections to refresh their understanding and the details of the algorithm design procedure. The focus in this section will be on how to construct the finite-element sets Λ_k.[2] As discussed in Section 3, constructing such sets is problem-specific and is the most challenging part in applying the design procedure to solve a specific optimization problem.

5.1. Algorithm Design

Phase 1. We choose x as a vector of locations of all base stations (denote x_m as the m-th component of x, $1 \le m \le M$). The objective function $f(x)$ here is the weighted capacity $\sum_{i=1}^{N} w_i r_i$. When x is given, $f(x)$ can be obtained by solving the following LP (polynomial complexity).

Maximize $\sum_{i=1}^{N} w_i r_i$
subject to

$$\sum_{1\le k\le N}^{k\ne i} f_{ki} + r_i - \sum_{1\le j\le N}^{j\ne i} f_{ij} - \sum_{m=1}^{M} f_{i,B_m} = 0 \qquad (1 \le i \le N)$$

[2] The notations used in this section are self-contained and do not relate to those in Section 4. For example, Λ_k's in this section are for the network capacity problem here and have no relationship to Λ_k's discussed in the last section for the network lifetime problem.

$$\sum_{1\le k\le N}^{k\neq i} \rho T f_{ki} + \sum_{1\le j\le N}^{j\neq i} c_{ij} T f_{ij} + \sum_{m=1}^{M} c_{i,B_m} T f_{i,B_m} \le e_i \quad (1 \le i \le N)$$

$$r_{\min} \le r_i \le r_{\max}, f_{ij}, f_{i,B_m} \ge 0 \quad (1\le i, j \le N, j \neq i, 1 \le m \le M) ,$$

where $r_{\min}$ and $r_{\max}$ denote the lower and upper bounds for the rate that a sensor can generate, respectively. Unlike the network lifetime problem in Section 4, now r_i are variables and T is a constant.

Phase 2. We now identify z_m, $\hat{g}_m(z_m)$, and $y_m^{(k)}$ for each x_m (the location of base station B_m). We choose z_m as a vector of $c_{i,B_m}T$ for $i = 1, 2, \cdots, N$. Denote z_{im} as the i-th component of z_m, i.e.,

$$z_{im} = c_{i,B_m} T .$$

For each z_{im}, we define

$$y_{im}^{(1)} = \theta_{i,B_m} ,$$

where θ_{i,B_m} is the corresponding phase of the base station B_m when the origin is sensor node i. For the rest of $y_{im}^{(k)}$ variables, we choose

$$y_{im}^{(2)} = e_{i,B_m}^t , \quad y_{im}^{(3)} = f_{i,B_m} , \quad L = 3 ,$$

and we can define z_{im} as

$$z_{im} = y_{im}^{(2)} \cdot (y_{im}^{(3)})^{-1} .$$

Similar to what we discussed in Section 4.2, it is sufficient to search only the locations that have $c_{i,B_m}T = \frac{e_{i,B_m}^t}{f_{i,B_m}}$, where e_{i,B_m}^t is the allocated transmission energy on link (i, B_m). We again omit the subscript im when there is no confusion.

Note that we must construct the finite-element sets Λ_2 and Λ_3 as geometric progressions, while Λ_1 does not have this requirement since y_1 is not used in the definition of z.

Claim *5.* **(Λ_1)** For $y_1 = \theta_{i,B_m}$ and an arbitrarily small given $\varepsilon_1 > 0$, we can construct a set $\Lambda_1 = \{h_1 a_1 : h_1 = 1, 2, \cdots, H_1\}$, with $H_1 = \lceil n\pi/\varepsilon_1 \rceil$ (where n is the path loss index) and $a_1 = 2\pi/H_1$ such that for any given solution ψ for base station placement, data routing, and energy allocation (on links) with a weighted capacity W, there exists a solution $\hat{\psi}$ and a sensor node i with $\theta_{i,B_m} \in \Lambda_1$ and the weighted capacity is $\hat{W} \ge (1 - \varepsilon_1)W$.

Proof. The proof is based on construction. That is, we will move base station B_m in solution ψ and construct $\hat{\psi}$ to satisfy all requirements.

Under solution ψ, for base station B_m, we consider $\frac{e_{j,B_m}^t}{f_{j,B_m}}$ for each sensor node j, $1 \le j \le N$. These $\frac{e_{j,B_m}^t}{f_{j,B_m}}$'s define a common region by intersecting disks from different node j. As discussed, we can move B_m to any boundary point of this region while the weighted capacity remains unchanged. For the purpose of this proof, we only consider moving base station B_m to a point on the arc (v_1, v_2) of the region's boundary that is part of the circle with the smallest radius (see Fig. 1 as an example). Assume the center of this circle is sensor node i and denote the point on this circle with phase ka_1 as w_k, $k = 1, 2, \cdots, H_1$. We now move B_m to a point w_k so that the new position for B_m has a phase ka_1 within the finite-element set Λ_1 while the weighted capacity will be decreased by no more than ε_1.

We consider two cases.

(i) We first consider the case that point w_k is on arc (v_1, v_2). After we move B_m to this point, under the same energy allocation, the new maximum allowed link rate of each link (i, B_m), which is $\frac{e_{i,B_m}^t}{c_{i,B_m}T}$, remains at least f_{i,B_m}, while the maximum allowed link rates of all other links remain unchanged. Therefore, the data routing in ψ is still feasible. Under the same data routing, the weighted capacity remains as W.

(ii) We now consider the case that none of these points w_k's is on arc (v_1, v_2) . In this case, we will move base station B_m to point w_k that is closet to this arc (see Fig. 1). We move base station B_m in two steps. In the first step, we move B_m to the end point v_1 of this arc toward w_k. Again this move will not change the weighted capacity. In the second step, we move B_m from vertex v_1 to w_k. As we discussed in the proof for Claim 1, for any sensor node j, the cost c_{j,B_m} can increase at most to $1+\varepsilon_1$. Therefore, under the same energy allocation in solution ψ, the maximum link rate of link (j, B_m) decreases at most to $1/(1 + \varepsilon_1) > 1 - \varepsilon_1$, while the maximum link rates of all other links remain unchanged. Therefore, the weighted capacity is at least $(1 - \varepsilon_1)W$.

Claim *6.* **(Λ_2)** For $y_2 = e_{i,B_m}^t$ and an arbitrarily small given $\varepsilon_2 > 0$, we can construct a set $\Lambda_2 = \{a_2 q_2^{h_2} : h_2 = 0, 1, \cdots, H_2\}$, where $a_2 = \varepsilon_2 e_i$, $q_2 = 1 + \varepsilon_2$, and $H_2 = \left\lfloor \frac{\ln(1/\varepsilon_2)}{\ln(1+\varepsilon_2)} \right\rfloor$, such that for any given solution ψ for base station placement, data routing, and energy allocation (on links) with a weighted capacity W, there exists a solution $\hat{\psi}$ with $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$, $\hat{e}_{i,B_m}^t \in \Lambda_1$ (when $\hat{e}_{i,B_m}^t > 0$), and the weighted capacity is $\hat{W} \ge (1 - \varepsilon_2)W$.

Proof. The proof is based on construction. That is, we will revise energy allocation in solution ψ and construct $\hat{\psi}$ to satisfy all requirements. Note that we keep the same base station placement, i.e., $\hat{\theta} = \theta$.

We can revise energy allocation in ψ and construct energy allocation in $\hat{\psi}$ by (7), (8), and (9). The feasibility for energy allocation in $\hat{\psi}$, i.e., the total allocated energy on each node i in $\hat{\psi}$ is no more than e_i, is proved in Claim 2.

We now show that the new weighted capacity is at least $(1 - \varepsilon_2)W$. Note that the maximum allowed link rate of link (i, B_m) is $\frac{e_{i,B_m}^t}{c_{i,B_m}T}$ and the maximum allowed link rate of link (i, j) is $\min\left\{\frac{e_{ij}^t}{c_{ij}T}, \frac{e_{ij}^r}{\rho T}\right\}$, we have: (1) when $0 < e_{i,B_m}^t < \varepsilon_2 e_i$, the maximum allowed link rate of link (i, B_m)

is increased; (2) when $e^t_{i,B_m} \geq \varepsilon_2 e_i$, the maximum allowed link rate of link (i, B_m) is decreased at most to $1/(1+\varepsilon_2) > 1-\varepsilon_2$; (3) for link (i, j) other than (i, B_m), the maximum allowed link rate is decreased to $1-\varepsilon_2$. Therefore, the maximum allowed link rate for each link in $\hat{\psi}$ is at least $(1-\varepsilon_2)$ of the maximum allowed link rate in ψ. To make data routing feasible, each link's rate as well as each node's rate need to decrease at most to $1-\varepsilon_2$. Therefore, the weighted capacity is at least $(1-\varepsilon_2)\cdot W$.

Claim 7. **(Λ_3)** For $y_3 = f_{i,B_m}$ and an arbitrarily small given $\varepsilon_3 > 0$, we can construct a set $\Lambda_3 = \{a_3 q_3^{h_3} : h_3 = 0, 1, \cdots, H_3\}$, with $a_3 = r_{\min}\varepsilon_3/2$, $q_3 = 1+\varepsilon_3/2$, and $H_3 = \left\lfloor \ln \frac{2Nr_{\max}}{\varepsilon_3 r_{\min}} / \ln\left(1+\frac{\varepsilon_3}{2}\right) \right\rfloor$, such that for any given solution ψ for base station placement and data routing with a weighted capacity W, there exists a solution $\hat{\psi}$ with $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$, $\hat{e}^t_{i,B_m} = e^t_{i,B_m}$, $f_{i,B_m} \in \Lambda_2$ when $f_{i,B_m} > 0$, and the weighted capacity is $\hat{W} \geq (1-\varepsilon_3)W$.

Proof. Again, the proof is based on construction. That is, we will revise the data routing in solution ψ and construct $\hat{\psi}$ to satisfy all requirements. Note that we keep $\hat{\theta}_{i,B_m} = \theta_{i,B_m}$ and $\hat{e}^t_{i,B_m} = e^t_{i,B_m}$.

This construction has two steps. In the first step, we construct a solution $\psi^\dagger$ from ψ with $f^\dagger_{i,B_m} \geq a_3$ when $f^\dagger_{i,B_m} > 0$ and the weighted capacity is $W^\dagger \geq (1-\varepsilon_3/2)W$. In the second step, we construct a solution $\hat{\psi}$ from $\psi^\dagger$ with $\hat{f}_{i,B_m} \in \Lambda_3$ when $\hat{f}_{i,B_m} > 0$ and the weighted capacity is $\hat{W} \geq (1-\varepsilon_3/2)W^\dagger > (1-\varepsilon_3)W$.

(i) We can revise the data routing in solution ψ and construct $\psi^\dagger$. Our objective is to ensure that in solution $\psi^\dagger$, $f^\dagger_{i,B_m} \geq a_3$ when $f^\dagger_{i,B_m} > 0$.

Since this revision will be done on flow level, we first decompose the data routing in ψ into flows as we discussed in the proof for Claim 3. We then revise the data routing for $\psi^\dagger$ by removing each flow f_{i,B_m} if $f_{i,B_m} < \frac{\varepsilon_3 r_{\min}}{2}$ for each node i. This removing is allowed, as node i's rate is a variable (i.e., adjustable). Based on Property 1, for each sensor node i, there is at most one $f_{i,B_m} > 0$ in ψ. Thus, we remove at most one flow from sensor i. Remove this flow will decrease node i's rate by $\frac{\varepsilon_3 r_{\min}}{2}/r_i \leq \varepsilon_3/2$. Therefore, $\psi^\dagger$ has the weighted capacity $W^\dagger \geq (1-\varepsilon_3/2)W$.

(ii) In the second step, we further revise the data routing in solution $\psi^\dagger$ and construct $\hat{\psi}$ as follows.

$$\hat{f}_{i,B_m} = \frac{\varepsilon_3 r_{\min}}{2}\left(1+\frac{\varepsilon_3}{2}\right)^{h_3} \quad (1 \leq i \leq N, 1 \leq m \leq M)$$

where $h_3 = \left\lfloor \ln \frac{2f^\dagger_{i,B_m}}{\varepsilon_3 r_{\min}} / \ln\left(1+\frac{\varepsilon_3}{2}\right) \right\rfloor$. Since $f^\dagger_{i,B_m} \leq \sum_{j=1}^N r_j \leq Nr_{\max}$, we have $h_3 \leq H_3 = \left\lfloor \ln \frac{2Nr_{\max}}{\varepsilon_3 r_{\min}} / \ln\left(1+\frac{\varepsilon_3}{2}\right) \right\rfloor$. That is, $\hat{f}_{i,B_m}$ is indeed within the set Λ_3.

Since the link rate of link (i, B_m) decreases at most to $1/(1+\varepsilon_3/2) > 1-\varepsilon_3/2$, to make data routing feasible, each node's rate need to decrease at most to $1-\varepsilon_3/2$. Therefore, $\hat{\psi}$ has weighted capacity $\hat{W} \geq (1-\varepsilon_3/2)W^\dagger \geq (1-\varepsilon_3/2)(1-\varepsilon_3/2)W > (1-\varepsilon_3)W$.

Phase 3. We now proceed to Phase 3. We first determine $\varepsilon_1, \varepsilon_2$, and ε_3, such that $\varepsilon_1+\varepsilon_2+\varepsilon_3 = \varepsilon$ and $q_2 = q_3 = q$. From Claims 6 and 7, $q_2 = 1+\varepsilon_2$ and $q_3 = 1+\varepsilon_3/2$, we choose $\varepsilon_1 = \varepsilon_2 = \varepsilon/4$ and $\varepsilon_3 = \varepsilon/2$.

For each $z = c_{i,B_m}T$, we have

$$\begin{aligned}
\Omega &= \{y_2 y_3^{-1} : y_k \in \Lambda_k, 2 \leq k \leq 3\} \\
&= \left\{ \varepsilon_2 e_i (1+\varepsilon_2)^{h_2} \left[\frac{\varepsilon_3 r_{min}}{2} \left(1+\frac{\varepsilon_3}{2}\right)^{h_3} \right]^{-1} \right\} \\
&= \left\{ \frac{\varepsilon e_i}{4} \left(1+\frac{\varepsilon}{4}\right)^{h_2} \left[\frac{\varepsilon r_{\min}}{4} \left(1+\frac{\varepsilon}{4}\right)^{h_3} \right]^{-1} \right\} \\
&= \left\{ \left(1+\frac{\varepsilon}{4}\right)^{h_2-h_3} \frac{e_i}{r_{\min}} \right\},
\end{aligned}$$

where $h_k = 0, 1, \cdots, H_k$, $k = 2, 3$, and

$$\begin{aligned}
|\Omega| &= O\left(\sum_{k=2}^{3} |\Lambda_k| \right) = O\left(\sum_{k=2}^{3} (H_k+1) \right) \\
&= O\left(\left\lfloor \frac{\ln(1/\varepsilon_2)}{\ln(1+\varepsilon_2)} \right\rfloor + \left\lfloor \ln \frac{2Nr_{\max}}{\varepsilon_3 r_{\min}} / \ln\left(1+\frac{\varepsilon_3}{2}\right) \right\rfloor + 2 \right) \\
&= O\left(\left\lfloor \frac{\ln(4/\varepsilon)}{\ln(1+\varepsilon/4)} \right\rfloor + \left\lfloor \ln \frac{4Nr_{\max}}{r_{\min}\varepsilon} / \ln\left(1+\frac{\varepsilon}{4}\right) \right\rfloor \right) \\
&= O\left(\frac{\ln(1/\varepsilon)}{\varepsilon} + \frac{\ln(N/\varepsilon)}{\varepsilon} \right) = O\left(\frac{\ln(N/\varepsilon)}{\varepsilon} \right).
\end{aligned}$$

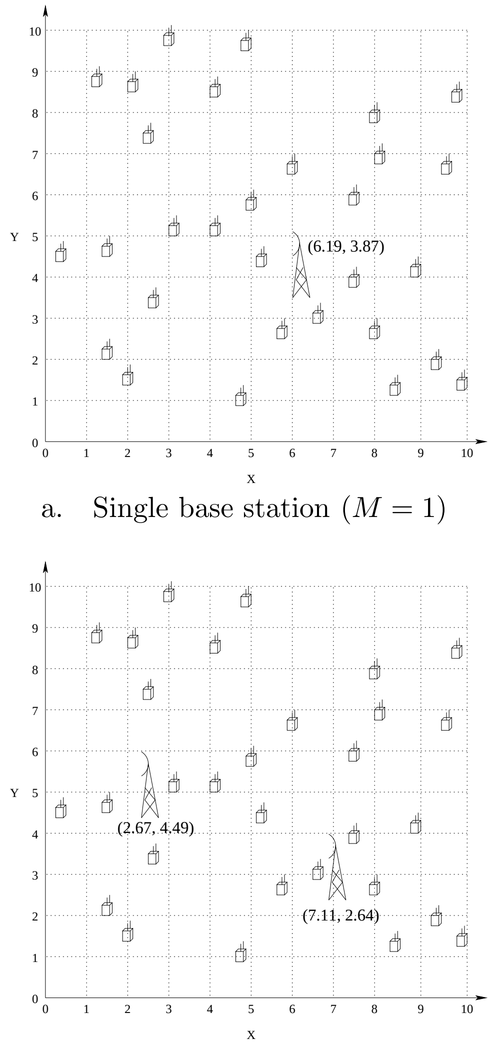
The set Γ for the locations of base station B_m is defined as all points with $\theta_{i,B_m} \in \Lambda_1$ and $c_{i,B_m}T \in \Omega$ (or $e_{i,B_m} \in \Lambda_2$ and $f_{i,B_m} \in \Lambda_3$), $1 \leq i \leq N$. Based on Claims 5, 6, and 7, we know that the maximum network longevity by checking all locations in Γ is at least $(1-\varepsilon)$ times the optimum and $|\Gamma| = O(N|\Omega||\Lambda_1|) = O(\frac{N}{\varepsilon^2}\ln\frac{N}{\varepsilon})$.

Phase 4. In Phase 4, we check all locations in Γ for each base station and find the maximum weighted capacity among them. Since there are M base stations, the search space is $O((\frac{N^2}{\varepsilon^2}\ln^2\frac{N}{\varepsilon})^M)$.

5.2. Numerical Examples

Again, we apply this $(1-\varepsilon)$ approximation algorithm to solve base station placement problem for $M = 1$ (single base station) and $M = 2$ (two base stations). We randomly generate a 30-node network in a 10x10 area (see Fig. 3). All units are normalized in consistent to those defined in (1), (2), and (3). For the power consumption model, we set $\alpha = 1$, $\beta = 3$, $\rho = 1$, and $n = 4$. The initial energy at a node is set from a uniform distribution within $[50, 100]$. The required network lifetime is 10 for all nodes. The weight for each node is set from a uniform distribution within $[1, 5]$. The minimum and maximum data rate are 1 and 100, respectively.

For a given $\varepsilon = 0.1$, the base station placements for $M = 1$ and $M = 2$ calculated by our approximation algorithm are shown in Figs. 3(a) and (b), respectively. The corresponding weighted network capacities are 3602.26 for $M = 1$ and 5767.96 for $M = 2$.



a. Single base station ($M = 1$)

b. Two base stations ($M = 2$)

Figure 3. Base station placement to maximize the weighted capacity.

6. Related Work

Due to energy constraint, the lifetime expectancy and network capacity for wireless sensor networks are limited. As a result, there is a flourish of research activities in this area in recent years. Many of these efforts (see, e.g., [1, 3, 9, 19] for network lifetime and [6, 10, 13, 15, 20] for network capacity) studied lifetime or capacity problems under given network topology and without explicit consideration on the impact of node placement on network performance.

Node placement problems in sensor networks include sensor node placement [4, 16, 17, 21], relay node placement [7, 18], and base station placement [2, 5, 11]. The main focus of sensor node placement has been on coverage in order to have either better geographical coverage of the area or better connectivity in the network. Relay node placement deals with how to place special auxiliary nodes within a sensor network so that network performance (e.g., connectivity, lifetime) can be improved. Related work on relay node placement (e.g., [7, 18]) have been limited on heuristic algorithm development without being able to provide performance guarantee.

Related work on base station placement include [2, 5, 11]. In [2], Bogdanov et al. studied how to place base station so that the network flow is proportionally maximized subject to link capacity. The authors show that although it is possible to find optimal solutions for special network topology (e.g., grid), the base station placement problem for an arbitrary network is NP-complete. The authors also pointed out that an approximation algorithm with any guarantee was not known and subsequently proposed two heuristic algorithms. In [11], Pan et al. studied single base station placement problem to maximize network lifetime (i.e., $M = 1$ case for our first problem). The optimal location is determined for the very special case when only single-hop routing between a sensor node and the base station is allowed. The more difficult problem for base station placement where multi-hop routing is allowed was not addressed.

The most relevant work to this paper is [5] by Efrat, Har-Peled, and Mitchell. In this work, the authors studied two location problems in sensor networks. The first problem addresses optimal location for a single base station placement, which is the same as the first problem discussed in this paper when $M = 1$. The authors proposed a $(1 - \varepsilon)$ approximation algorithm that has $O\left(\frac{N}{\varepsilon^4} \ln \frac{N}{\varepsilon}\right)$ computational complexity. In comparison, for single base station placement ($M = 1$), the computational complexity in the approximation algorithm developed in this paper is $O\left(\frac{N}{\varepsilon^2} \ln \frac{N}{\varepsilon}\right)$, which is order of $1/\varepsilon^2$ reduction in complexity. Such reduced complexity is mainly attributed to our development of the complexity reduction technique discussed in Section 3.2. More important, we have made a theoretical contribution by synthesizing a systematic design procedure in Section 3.2, which has the potential to be applied for the design of $(1 - \varepsilon)$ approximation algorithms to solve a broader class of problems.

7. Conclusions

Our efforts in this work were motivated by base station placement problems in sensor networks. Prior to this work, there was only one $(1 - \varepsilon)$ approximation algorithm for base station placement but unfortunately with high complexity. In this paper, we developed a procedure to design $(1 - \varepsilon)$ approximation algorithms that not only produce an approximation algorithm with much lower complexity, but also can be applied to address other difficult problems for base station placement with other objectives (i.e., network capacity). The proposed procedure offers a general framework to the design of $(1 - \varepsilon)$ approximation. The key ideas are to transform infinite search space to a finite-element search space with performance guarantee and to exploit overlap among the elements to further reduce the size of the search space. We believe this approach has the potential to solve other difficult optimization problems involving continuous search space and we are currently further exploring its applications beyond the two problems discussed in this paper.

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Appendix

Lemma *4.* For the maximum network longevity problem discussed in Section 4, if the search space for the locations of the M base stations is limited to the locations of the N sensor nodes, then the maximum network longevity T_S is at least $2^{-n}T^*$, where T^* is the maximum network longevity.

Proof. We first prove the following result. For any given solution ψ for base station placement, data routing, and energy allocation (on links) with the network longevity T, there exists a solution $\hat{\psi}$, where each base stations is located at the same place of a sensor node, with the network longevity $\hat{T} \geq 2^{-n}T$.

For each base station B_m, $1 \leq m \leq M$, we move it from its location in solution ψ to its nearest sensor node, which we denote as $\hat{B}_m$. We call this newly constructed solution as $\hat{\psi}$. For each sensor node i, $1 \leq i \leq N$, we have $d_{i,\hat{B}_m} \leq d_{i,B_m} + d_{B_m,\hat{B}_m}$ and $d_{B_m,\hat{B}_m} \leq d_{i,B_m}$ (otherwise, base station B_m will be relocated to sensor node i). Thus, $d_{i,\hat{B}_m} \leq 2d_{i,B_m}$. By (2), we have $c_{i,\hat{B}_m} = \alpha + \beta \hat{d}^n_{i,B_m} \leq \alpha + \beta(2d_{i,B_m})^n < 2^n\alpha + \beta(2d_{i,B_m})^n = 2^n c_{i,B_m}$. The link longevity for link (i, B_m) decreases at most by a factor of 2^{-n}, i.e., the network longevity under $\hat{\psi}$ is at least $\hat{T} \geq 2^{-n}T$.

For the special case that $\psi = \psi^*$, i.e., an optimal solution, with the network longevity T^*, we know that there exists a solution $\hat{\psi}^*$, where each base stations is re-located to the location of its nearest sensor with a network longevity $\hat{T}^* \geq 2^{-n}T^*$. Note that $T_S \geq \hat{T}^*$, we have $T_S \geq 2^{-n}T^*$.

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