Toward Simple Criteria to Establish Capacity Scaling Laws for Wireless Networks

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Abstract

Capacity scaling laws offer fundamental understanding on the trend of user throughput behavior when the network size increases. Since the seminal work of Gupta and Kumar, there have been active research efforts in developing capacity scaling laws for ad hoc networks under various advanced physical layer technologies. These efforts led to different custom-designed solutions, most of which were intellectually challenging and lacked universal properties that can be extended to address scaling laws of ad hoc networks with other physical layer technologies. In this paper, we present a set of simple yet powerful general criteria that one can apply to quickly determine the capacity scaling laws for various physical layer technologies under the protocol model. We prove the correctness of our proposed criteria and demonstrate their usage through a number of case studies, such as ad hoc networks with directional antenna, MIMO, multi-channel multi-radio, cognitive radio, and multiple packet reception. These simple criteria will serve as powerful tools to networking researchers to obtain throughput scaling laws of ad hoc networks under different physical layer technologies, particularly those to be developed in the future.

Keywords

Asymptotic capacity, scaling law, physical layer technology

1 Introduction

Capacity scaling laws refer to how a user's throughput scales as the network size increases to infinity.[1] Such scaling law results, expressed in $O(\cdot)$, $\Omega(\cdot)$, and $\Theta(\cdot)$ as a function of n (where n is the number of nodes in the network and

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[1]When there is no ambiguity, we use the terms "asymptotic capacity" and "capacity scaling law" interchangeably throughout this paper.

approaches infinity), offer fundamental understanding on the trend of user throughput behavior when the network size increases.

Since the seminal results of Gupta and Kumar ("G&K" for short) on capacity scaling law of ad hoc networks with classical single omnidirectional antenna [4], there has been a flourish of research efforts on exploring capacity scaling laws for ad hoc networks under various physical layer technologies. These include directional antennas [11, 20], MIMO [7], multi-channel multi-radio (MC-MR) [8], cognitive radios [5, 6, 14, 21], and multiple packet reception (MPR) [12], among others. For each of these advanced physical layer technologies, a custom-designed analytical approach was employed to develop its capacity scaling law. Each of these solutions is usually intellectually challenging and lacks universal properties that can be extended to address scaling laws of ad hoc networks with other physical layer technologies.

A fundamental question we ask in this paper is the following: *instead of custom-designing a sophisticated analytical approach for each physical layer technology, can we devise a set of simple yet universal rules (or general criteria) that one can easily apply to quickly determine the capacity scaling law for various physical layer technologies?* Should such unified rules/criteria exist, then they will offer a set of powerful tools to networking researchers to understand throughput scaling behavior of ad hoc networks under various physical layer technologies, particularly those new technologies that will appear in the future.

The main contribution of this paper is the development of simple criteria for establishing capacity upper bounds under the protocol model for ad hoc networks under various physical layer technologies. The following is a summary of our contributions.

- We conceive a novel "interference square" concept that divides a normalized 1×1 network area into small interference squares, each with side length $1/\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \rceil$, where $r(n)$ is the transmission range and Δ is a parameter to set the interference range under the protocol model. We offer some unique interference properties regarding transmissions inside each interference square.

- Based on the new interference square concept, we develop two simple yet powerful scaling order criteria to determine the asymptotic capacity upper bounds for various physical layer technologies. Either criterion is sufficient to give a capacity upper bound for a given physical layer technology, and the choice to use which criterion is purely a matter of convenience and only depends on the underlying problem. We also prove the correctness of applying these criteria to obtain capacity upper bounds.

- To demonstrate the usage of our criteria, we study asymptotic capacity of ad hoc networks under various physical layer technologies, such as directional antenna, MIMO, MC-MR, cognitive radio, and MPR. We show that by applying our simple criteria, one can easily obtain capacity upper bounds under these physical layer technologies, which are consistent to those results in the literature that were developed under custom-designed analytical approaches. Note that our criteria not only can recover those results already reported in literature, but can also determine the upper bounds of ad hoc networks with certain physical layer technology that has not been studied before, and ad hoc networks with new physical layer technologies that will appear in the future.

The only limitation of our simple criteria is that it is designed to derive capacity upper bounds. For capacity lower bounds, we argue that a set of simple criteria do not appear to exist, and we give rational on why this is the case in Section 10.

The remainder of this paper is organized as follows. In Section 2, we take a closer look at G&K's approach (for ad hoc networks with classical single omnidirectional antennas) and understand why it falls short as an universal approach for various physical layer technologies. Subsequently, in Section 3, we propose a novel interference square concept and based on this concept, in Section 4, we present two simple yet powerful scaling order criteria, which can be used to quickly derive capacity upper bounds for various physical layer technologies. To demonstrate our criteria, in Sections 5 to 9, we apply our simple criteria to ad hoc networks based on different physical layer technologies such as directional antenna, MIMO, MC-MR, cognitive radio, and MPR. We show that one can easily obtain capacity upper bounds for these networks, which are consistent to those reported in the literature under custom-designed analysis. Section 10 offers some discussions of our work and Section 11 concludes this paper. Table 1 lists notation used in this paper.

2 Lesson Learned From G&K's Approach

In this section, we take a close look at G&K's approach in analyzing capacity scaling law and try to understand why such an approach poses a barrier in analyzing capacity scaling laws when advanced physical layer technologies are employed.

2.1 Background

In G&K's work [4], they considered an ad hoc network with n nodes that are randomly located within a unit square area. Each node in the network is a source node and transmits its data to a randomly chosen destination node. A node's transmission is limited by its transmission range. When the distance between a source node and its destination node is large, multi-hop routing is needed to relay the data. The per-node throughput $\lambda(n)$ is defined as the data rate that can be sent from each source to its destination. A capacity scaling law attempts to characterize the maximum per-node throughput $\lambda(n)$ when the number of nodes n goes to infinity.

In [4], two interference models, the protocol model and the physical model, were considered in their study. In this study, we focus on the protocol model and leave the physical model for future research. In the protocol model [4], each transmitting node is associated with a transmission range $r(n)$, and an interference range $(1+\Delta)r(n)$,

Table 1: Notation.

General notation	
d_{ij}	Distance between nodes i and j
D	Average distance between all source-destination pairs
$f_{\text{RX}}(n)$	An upper bound for the maximum number of successful transmissions whose receivers are in the same interference square
$f_{\text{TX}}(n)$	An upper bound for the maximum number of successful transmissions whose transmitters are in the same interference square
n	The number of nodes in the network
$\mathcal{N}$	The set of nodes in the network
W	The data rate of a successful transmission in a channel
$r(n)$	The (common) transmission range of all nodes under the protocol model
$\text{Rx}(l)$	Receiver of link l
$\text{Tx}(l)$	Transmitter of link l
Δ	A parameter to set interference range in the protocol model
$\lambda(n)$	Per-node throughput of a random network with n nodes
Ad hoc network with directional antennas	
S	An interference square in the unit area
A_S	Area of S
N_S	Number of nodes in S
MIMO ad hoc network	
$\mathcal{I}_l$	The set of links that are interfered by link l
$\mathcal{Q}_l$	The set of links that are interfering link l
z_l	Number of data streams on link l
α	Number of antennas at each node
$\Pi(\cdot)$	The mapping between a node and its order in the node list
MC-MR network	
c	The number of channels in the network
m	The number of radio interfaces at each node
CR ad hoc network	
$\mathcal{B}_i$	The set of available bands at node i
$\mathcal{B}_{ij}$	The set of available bands on link (i, j)
M	$= \lvert \bigcup_{i=1}^{n} \mathcal{B}_i \rvert$, i.e., the number of distinct frequency bands in the network
Ad hoc network with MPR	
β_1	Number of simultaneous packets from intended transmitters whose transmission range covers a receiver
β_2	Number of unintended transmitters that produce interference on the same receiver
β	A constant representing the total available resource at a receiver

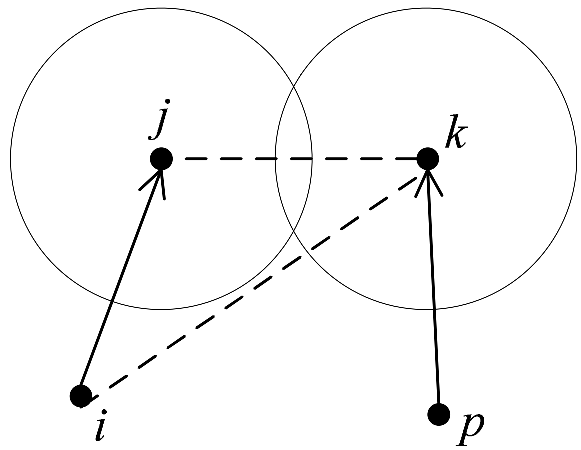


Figure 1: Overlapping of two circular footprints of two receiving nodes.

where Δ is a constant. To guarantee the connectivity of the network, transmission range $r(n)$ must satisfy the following condition (regardless of the underlying physical layer technology) [3]:

$$r(n) \geq \sqrt{\frac{\ln n}{n}} \, . \tag{1}$$

When node i transmits to node j, the necessary and sufficient conditions for a successful transmission are

- node j is within the transmission range of node i, i.e., $d_{ij} \leq r(n)$, where d_{ij} is the distance between nodes i and j, and
- node j is outside the interference range of any other transmitting node k, i.e., $d_{kj} > (1 + \Delta)r(n)$, $k \neq i$.

In [4], when the transmission from a node to another node is successful, then the achieved data rate for this transmission is assumed to be a constant W.

2.2 G&K's Approach and Its Limitation

A key component in G&K's approach in deriving capacity upper bound is to calculate how much footprint area each successful transmission occupies. Then by dividing the unit square area by this area, they were able to obtain an upper bound of the maximum number of successful transmissions at a time and subsequently to derive a capacity upper bound. Specifically, in [4], G&K showed that for successful reception at each receiver, one can draw a circle around each receiver with radius $\frac{\Delta r(n)}{2}$ and these circles must be disjoint. This result can be proved by contradiction. That is, suppose two two circles centered at receivers j and k with radius $\frac{\Delta r(n)}{2}$ are not disjoint (see Fig. 1), then $d_{jk} \leq \Delta r(n)$. Suppose receiver j is receiving data from transmitter i. Then we have $d_{ij} \leq r(n)$. Based on the

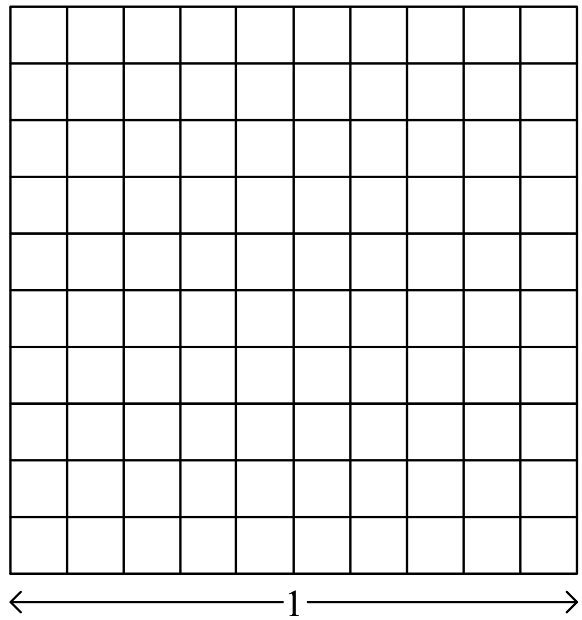


Figure 2: The unit square is divided into equal-sized small squares. Each small square has a side length of $1/\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \rceil$.

triangle inequality, we have $d_{ik} \leq d_{ij} + d_{jk} \leq (1 + \Delta)r(n)$, which means that receiver k is within the interference range of i. But this contradicts with the fact that receiving node k must fall out of the interference range of node i. Under the above approach, a successful transmission will occupy a footprint area of $\pi \left[\frac{\Delta r(n)}{2}\right]^2$. Then the maximum number of successful transmissions within the unit square area is at most $1/[\pi(\frac{\Delta r(n)}{2})^2]$ at any time. Based on this result, G&K derived a capacity upper bound.

The essence of the above footprint area approach is to identify the size of the circular area that each successful transmission will occupy. But this approach poses a barrier when we encounter advanced physical layer technologies (e.g., MIMO, directional antennas) beyond single omnidirectional antenna node considered in [4]. This is because under these advanced physical layer technologies, the interference relationships among the nodes are much more complicated than that under the single omnidirectional antenna scenario in [4]. As a result, the footprint area of each successful receiver does *not* have to be disjoint. For example, in a MIMO ad hoc network where each node employs multiple transmit/receive antennas, receiving node k in Fig. 1 may use its degree-of-freedoms (DoFs) to cancel the interference from transmitting node i [1, 15]. As a result, G&K's approach of associating disjoint footprint area with each successful transmission falls apart here.

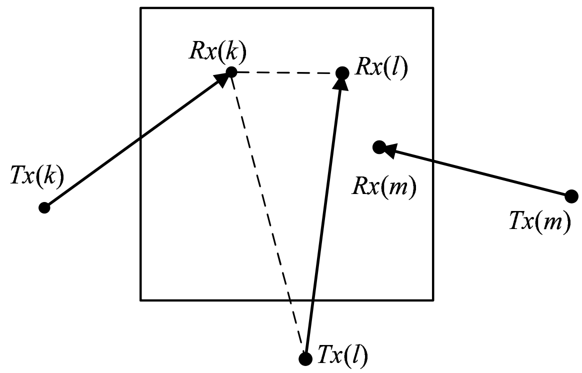


Figure 3: A set of transmissions whose receivers are in the same interference square.

3 A New Approach

Given that the footprint area approach in [4] is not capable of handling more complex interference relationships (brought by advanced physical layer technologies), we propose a new approach that handles interference from a different perspective. *Instead of focusing on how much footprint area each successful transmission occupies, we will calculate how many successful transmissions that a given small area in the network can support.* Specifically, we divide the unit square into small equal-sized squares (Fig. 2) with the side length of each small square being $1/\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \rceil$. We call each small square an *interference square.* As we shall show in Section 4, if one can find the maximum number of successful transmissions in each interference square (under a specific physical layer technology), then we can derive the capacity upper bound for the entire network. Subsequently, in Sections 5 to 9, we show how to find the maximum number of successful transmissions in each interference square under different physical layer technologies, thus deriving capacity upper bound for each of these technologies.

Before we show how this new interference square approach can offer simple scaling law criteria, we discuss some important properties associated with such small squares as follows.

Property 1 *For a set of successful simultaneous transmissions whose receivers fall in the same interference square, the receiver of any such transmission must be within the interference range of any other transmitter from the same set of transmissions.*

Proof Note that the distance between any two receivers in the same interference square is at most $\sqrt{2} \cdot \frac{\Delta r(n)}{\sqrt{2}} = \Delta \cdot r(n)$. Denote $\text{Tx}(l)$ and $\text{Rx}(l)$ the transmitter and receiver of transmission l, respectively. Referring to Fig. 3, for any two transmissions l and k with their receivers $\text{Rx}(l)$ and $\text{Rx}(k)$ in the interference square, we have $d_{\text{Rx}(l),\text{Rx}(k)} \leq \Delta \cdot r(n)$. Since $d_{\text{Tx}(l),\text{Rx}(l)} \leq r(n)$ (recall that $r(n)$ is transmission range) based on the triangle inequality, we have

$d_{\text{Tx}(l),\text{Rx}(k)} \leq d_{\text{Rx}(l),\text{Rx}(k)} + d_{\text{Tx}(l),\text{Rx}(l)} \leq (1+\Delta)r(n)$. Similarly, we can prove that the receiver $\text{Rx}(l)$ of transmission l is also in the interference range of transmitter $\text{Tx}(k)$ of transmission k. □

Similar to Property 1 (which considers receivers in the same interference square), we can consider transmitters in the same interference square and have the following property.

Property 2 *For a set of successful simultaneous transmissions whose transmitters reside in the same interference square, the receiver of any such transmission must be within the interference range of any other transmitter from the same set of transmissions.*

The proof of Property 2 is similar to that of Property 1 and is omitted.

Properties 1 and 2 show us two complementary ways on how to assess interference relationship when considering either receivers or transmitters in the same interference square. It turns out that these two properties allow us to calculate the number of successful transmissions with either their receivers or transmitters in the same interference square under various physical layer technologies. For example, under the single omnidirectional antenna setting in Section 2.1, we can easily conclude that there can be at most one active receiver (or transmitter) in an interference square for a successful transmission and the maximum number of successful transmissions with either receivers or transmitters in the same interference square is one. As another example, for MIMO ad hoc network where each node is equipped with multiple transmit/receiver antennas, Properties 1 and 2 allow us to show that the maximum number of successful transmissions whose receivers (or transmitters) in the same interference square is upper bounded by the number of antennas at each node (see details in Section 6). As we shall show in the next section (Theorems 1 and 2), the maximum number of successful transmissions whose receivers (or transmitters) are in the same interference square will determine the capacity scaling law of an ad hoc network under various advanced physical layer technologies.

4 Main Results: Simple Scaling Order Criteria

As we shall show in Sections 5 to 9, for a specific physical layer technology, the newly defined interference square and Properties 1 and 2 enable us to characterize the maximum number of successful transmissions whose receivers (or transmitters) are in the same interference square. For a specific physical layer technology, denote $f_{\text{RX}}(n)$ as an upper bound for the maximum number of successful transmissions whose receivers are in the same interference square. Similarly, denote $f_{\text{TX}}(n)$ as an upper bound for the maximum number of successful transmissions whose transmitters are in the same interference square. In this section, we show that once we have $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, we can quickly determine a capacity scaling order based on either one of two simple scaling order criteria. Figure 4 summarizes the idea of the above discussion.

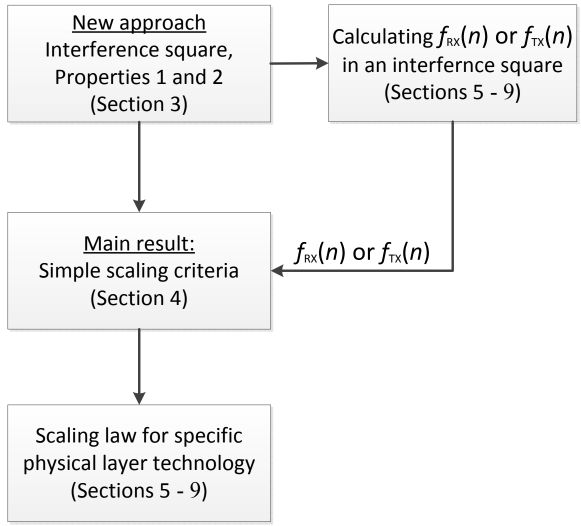


Figure 4: A flowchart illustrating our approach to derive capacity scaling law for a specific physical layer technology.

The two criteria that we present in this section (Theorem 1 and 2) show that the capacity upper bound scales asymptotically with either $\frac{f_{\rm RX}(n)}{nr(n)}$ or $\frac{f_{\rm TX}(n)}{nr(n)}$. We formally state these results as follows.

Theorem 1 (Criterion 1) *For a given $f_{RX}(n)$, the asymptotic capacity of a random ad hoc network is*

$$\lambda(n) = O\left(\frac{f_{RX}(n)}{nr(n)}\right)$$

almost surely when $n \to \infty$. In the special case when $f_{RX}(n)$ is a constant, then $\lambda(n) = O(1/\sqrt{n \ln n})$ almost surely when $n \to \infty$.

The proof of Theorem 1 is given in the appendix. Similarly, if we can find $f_{\rm TX}(n)$, then the following criterion can also give an upper bound for the asymptotic capacity.

Theorem 2 (Criterion 2) *For a given $f_{TX}(n)$, the asymptotic capacity of a random ad hoc network is*

$$\lambda(n) = O\left(\frac{f_{TX}(n)}{nr(n)}\right)$$

almost surely when $n \to \infty$. In the special case when $f_{TX}(n)$ is a constant, then $\lambda(n) = O(1/\sqrt{n \ln n})$ almost surely when $n \to \infty$.

The proof of Theorem 2 is similar to that of Theorem 1 and is omitted to conserve space.

Several remarks about the above two criteria are in order. First, for a specific physical layer technology, we only need to focus on the calculation of either $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, whichever is more convenient. An asymptotic capacity will follow once we have either $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, based on either Theorem 1 or Theorem 2. Second, when either $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$ is a constant, then the asymptotic capacity upper bound is $O(1/\sqrt{n \ln n})$, which is precisely the same as that in [4] by G&K for the protocol model. This offers a quick test on whether the underlying physical layer technology will indeed change the scaling order of capacity upper bound comparing to the classical single omnidirectional antenna based ad hoc network in [4]. Finally, the two criteria allow us to focus on calculation ($f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$) only within a small interference square. The details associated with network-wide multi-hop end-to-end throughput have been folded in the proof of the two theorems and are no longer of concerns to users of these two theorems in deriving asymptotic capacity upper bound for a given physical layer technology.

Example 1 As the first application of our scaling order criterion, let's validate the classical single omnidirectional antenna based ad hoc network considered in [4]. As discussed in Section 3, we have that $f_{\text{RX}}(n) = 1$. Thus, by Theorem 1, we have $\lambda(n) = O(1/\sqrt{n \ln n})$, which is precisely the same result in [4] by G&K. □

In the remaining several sections, we will explore capacity scaling laws for ad hoc networks under various physical layer technologies. Referring to Fig. 4, for each case, we will first calculate $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, whichever is more convenient, based on the new interference square and Properties 1 and 2. This is the upper righthand block in Fig. 4. Once we have $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, then we will apply one of the two criteria in this section to quickly obtain the capacity scaling law for this physical layer technology (bottom block in Fig. 4).

5 Case Study I: Ad Hoc Networks with Directional Antennas

Compared to omnidirectional antenna, directional antenna can control its beam width and concentrate its beam toward its intended destination. Since nodes outside the beam is not interfered, greater spatial reuse inside the network can be achieved. In this section, we apply our criteria in Section 4 to explore asymptotic capacity of a random ad hoc network with each node being equipped with a directional antenna. We follow the same model as in [11] by Peraki and Servetto.[2] The scaling law results in [11] are well known and widely cited. They showed that for single-beam model, the asymptotic capacity scales as $O\left(r(n)\right)$ and for multi-beam model, it scales as $O\left(nr^3(n)\right)$. The analysis approach in [11] was custom-designed and differed from that by G&K. In this section, we show that by applying our criteria in Section 4, we can quickly obtain the same results for asymptotic capacity upper bound in [11].

[2]Another major work on scaling law for directional antennas is [20] by Yi *et al.*, which employed a slightly different model and thus led to a different set of results. The approach in [20] followed the same token as that in [4] by G&K. It can be shown that our criteria can be easily applied there and we leave the details to readers as an exercise.

We organize this section as follows. First, we consider the case for the single-beam model. Then, we consider the multi-beam model.

5.1 Scaling Law Analysis for Single Beam Model

5.1.1 Single Beam Model

In [11], single beam model refers that a transmitter can generate at most one directional beam to an intended receiver, although a receiver can receive multiple directed beams from different nodes.

5.1.2 Calculating $f_{\text{TX}}(n)$

In this case study, we choose to calculate $f_{\text{TX}}(n)$, which is more convenient than $f_{\text{RX}}(n)$. As discussed in Section 4, the choice of calculating $f_{\text{TX}}(n)$ or $f_{\text{RX}}(n)$ is solely based on convenience and either one is sufficient to determine asymptotic capacity.

Recall that $f_{\text{TX}}(n)$ is an upper bound for the maximum number of successful transmissions whose transmitters are in the same interference square. In the case of single-beam model, $f_{\text{TX}}(n)$ corresponds to an upper bound for the maximum number of successful beam transmission whose transmitters are in the same interference square. To calculate $f_{\text{TX}}(n)$, we need the following lemma.

Lemma 1 *The number of nodes in the same interference square is $\Theta(nr^2(n))$ almost surely when $n \to \infty$.*

Proof Denote S an interference square within the unit area. Denote A_S and N_S the area and the number of nodes in S, respectively. Since nodes in S are randomly distributed, we have the average number of nodes in S is $E(N_S) = nA_S$. For the number of nodes in S, we have the following probabilities (also known as Chernoff bounds) [10].

- For any $\delta > 0$, $P\{N_S > (1+\delta)nA_S\} < \left[\frac{e^\delta}{(1+\delta)^{1+\delta}}\right]^{nA_S}$.
- For any $0 < \delta < 1$, $P\{N_S < (1-\delta)nA_S\} < e^{-\frac{1}{2}nA_S\delta^2}$.

Combining the above two inequalities, for any $0 < \delta < 1$, we have

$$
\begin{aligned}
& P\{|N_S - nA_S| > \delta nA_S\} \\
=& P\{N_S > (1+\delta)nA_S\} + P\{N_S < (1-\delta)nA_S\} \\
<& \left[\frac{e^\delta}{(1+\delta)^{1+\delta}}\right]^{nA_S} + e^{-\frac{1}{2}nA_S\delta^2} \\
=& e^{-\theta_1 nA_S} + e^{-\theta_2 nA_S}\ , \qquad (2)
\end{aligned}
$$

where $\theta_1 = (1+\delta)\ln(1+\delta) - \delta$ and $\theta_2 = \frac{1}{2}\delta^2$.

Note that $A_S = \frac{1}{\left\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \right\rceil^2} = \Theta(r^2(n))$. Letting $A_S = \Theta(r^2(n))$ in (2), we have

$$P\left\{|N_S - nA_S| > \delta n A_S\right\} < e^{-\theta_1 n \Theta(r^2(n))} + e^{-\theta_2 n \Theta(r^2(n))} \,. \tag{3}$$

Based on (1), we have $r(n) = \Omega(\sqrt{\frac{\ln n}{n}})$. Thus, the right-hand-side of (3) goes to zero when $n \to \infty$, which shows that the probability that the deviation of the number of nodes in S from the mean by more than a constant factor of the mean is zero when $n \to \infty$. Based on the definition of $\Theta(\cdot)$, we have $N_S = \Theta(nr^2(n))$. □

Based on Lemma 1, we have the following lemma for $f_{\text{TX}}(n)$.

Lemma 2 For a random ad hoc network under single-beam directional antenna, we have $f_{\text{TX}}(n) = \Theta(nr^2(n))$.

Proof By Lemma 1, there are $\Theta(nr^2(n))$ nodes in the interference square. Since each node can only generate one beam, the total number of successful beam transmissions generated by the transmitters in this interference square cannot exceed $\Theta(nr^2(n))$, i.e., $f_{\text{TX}}(n) = \Theta(nr^2(n))$. □

5.1.3 Scaling Law

Following Fig. 4, with $f_{\text{TX}}(n) = \Theta(nr^2(n))$, we can now apply Theorem 2 and quickly obtain the following capacity scaling law.

Proposition 1 *For a random ad hoc network under single-beam directional antenna, we have* $\lambda(n) = O\left(r(n)\right)$ *almost surely when* $n \to \infty$.

Proof Combining Lemma 2 and Theorem 2, we have $\lambda(n) = O\left(\frac{f_{\text{TX}}(n)}{nr(n)}\right) = O\left(nr^2(n) \cdot \frac{1}{nr(n)}\right) = O\left(r(n)\right)$. □

Note that this result for single-beam case is the same as that in [11].

5.2 Scaling Law Analysis for the Multi-Beam Model

5.2.1 Multi-Beam Model

In [11], multi-beam model refers that a transmitting node can generate multiple beams to different receiving nodes at the same time. On the other hand, a receiving node can only receive one beam from the same transmitting node but may receive multiple beams from different transmitting nodes. We follow the same model in [11] for the multi-beam case.

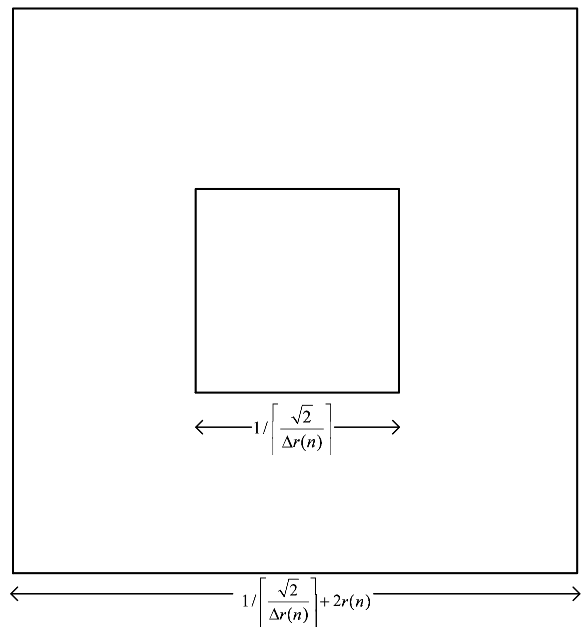


Figure 5: The larger square contains all the transmitters that can transmit directional beams to the receivers that are in the small interference square at the center.

5.2.2 Calculating $f_{\mathbf{RX}}(n)$

We will calculate $f_{\text{RX}}(n)$.[3] Recall that $f_{\text{RX}}(n)$ is an upper bound of the maximum number of successful transmissions whose receivers are in the same interference square. In the case of multi-beam model, $f_{\text{RX}}(n)$ corresponds to an upper bound of the maximum number of successful beam transmissions received by the receivers that are in the same interference square.

For receivers residing in the same interference square, it is easy to see that their transmitters cannot be outside a larger square, with the same center as the interference square, but with side length of $1/\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil + 2r(n)$ (see Fig. 5). Otherwise, a receiver in the interference square will be outside of a transmitter's transmission range $r(n)$. For the number of nodes inside the larger square (regardless of transmitters or receivers), we have the following lemma.

Lemma 3 *The number of nodes in the larger square with side length* $2r(n) + \frac{1}{\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil}$ *is* $\Theta(nr^2(n))$ *almost surely when* $n \to \infty$.

The proof of Lemma 3 is similar to the proof of Lemma 1 and is omitted here. Now, we are ready to calculate $f_{\text{RX}}(n)$ as follows.

Lemma 4 *For a random ad hoc network under multi-beam directional antenna, we have* $f_{RX}(n) = O(n^2r^4(n))$.

[3] The level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as $f_{\text{TX}}(n)$ for the multi-beam model. Either choice will lead to the same result.

Proof Based on Lemma 3, we know that the number of transmitters that can transmit beams to the same receiver in the interference square is at most $O(nr^2(n))$. That is, a receiver in the interference square can receive at most $O(nr^2(n))$ beams. By Lemma 1, there are at most $\Theta(nr^2(n))$ receivers in the same interference square. So we have $f_{\text{RX}}(n) = \Theta(nr^2(n)) \cdot O(nr^2(n)) = O(n^2r^4(n))$. □

5.2.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = O(n^2r^4(n))$, we can now apply Theorem 1 and quickly obtain the following capacity scaling law.

Proposition 2 *For a random ad hoc network under multi-beam directional antenna, we have $\lambda(n) = O\left(nr^3(n)\right)$ almost surely when $n \to \infty$.*

Proof Combining Lemma 4 and Theorem 1, we have $\lambda(n) = O\left(\frac{f_{\text{RX}}(n)}{nr(n)}\right) = O\left(n^2r^4(n) \cdot \frac{1}{nr(n)}\right) = O\left(nr^3(n)\right)$.
□

This result is the same as that in [11] for the multi-beam case.

6 Case Study II: MIMO Ad Hoc Networks

6.1 MIMO Model

By employing multiple antennas at both transmitting and receiving nodes, MIMO has brought significant benefits to wireless communications, such as increased link capacity [2, 16], improved link diversity [23], and interference cancellation between conflicting links [1, 15]. In this section, we characterize asymptotic capacity for multi-hop MIMO ad hoc networks. Although there are many schemes to exploit the benefits of antenna arrays at a node, we focus on the two key characteristics of MIMO: *spatial multiplexing* and *interference cancellation* [1, 15, 22]. Spatial multiplexing refers that a transmitter can send several independent data streams to its intended receiver simultaneously on a link. Interference cancellation refers that by properly exploiting multiple antennas at a node, potential interference to and/or from other nodes can be cancelled.

To model spatial multiplexing and interference cancellation, we employ recent advance in MIMO link model in [13] by Shi *et al.* In this model, degree-of-freedom (DoF) is used to represent resource at a MIMO node. Simply put, the number of DoFs at a node is equal to the number of antennas, denoted as α, at the node. Denote z_l the number of active data streams on link l in a time slot. Denote Tx(l) and Rx(l) the transmitter and the receiver of link l, respectively. To spatial multiplex z_l data streams on link l, we need to allocate z_l ($z_l \leq \alpha$) DoFs at both transmitter Tx(l) and receiver Rx(l). To cancel interference from and/or to other nodes in the network, it is necessary to have an ordered list for all nodes and allocate DoFs at each node following this order [13]. Denote $\Pi(\cdot)$ the mapping between a node and its order in the node list. Suppose that link l is carrying z_l data streams. Denote $\mathcal{I}_l$ and $\mathcal{Q}_l$ the set of links that are interfered by link l and the set of links that are interfering link l, respectively. Transmitter $\text{Tx}(l)$ is responsible for cancelling the interference from itself to all receivers $\text{Rx}(k)$, $k \in \mathcal{I}_l$, that are before node $\text{Tx}(l)$ in the order list. Similarly, receiver $\text{Rx}(l)$ of link l is responsible for cancelling the interference from all transmitters $\text{Tx}(k)$, $k \in \mathcal{Q}_l$, that are before node $\text{Rx}(l)$ in the order list. Since the total number of DoFs for spatial multiplexing and interference cancellation cannot exceed α, we have the following two constraints on each active link l in the network.

1. DoF constraint at $\text{Tx}(l)$: The number of DoFs that $\text{Tx}(l)$ can use for spatial multiplexing (for transmission) and interference cancellation cannot exceed the total number of DoFs at node $\text{Tx}(l)$, i.e.,

$$z_l + \sum_{k \in \mathcal{I}_l}^{\Pi(\text{Tx}(l)) > \Pi(\text{Rx}(k))} z_k \leq \alpha \,. \tag{4}$$

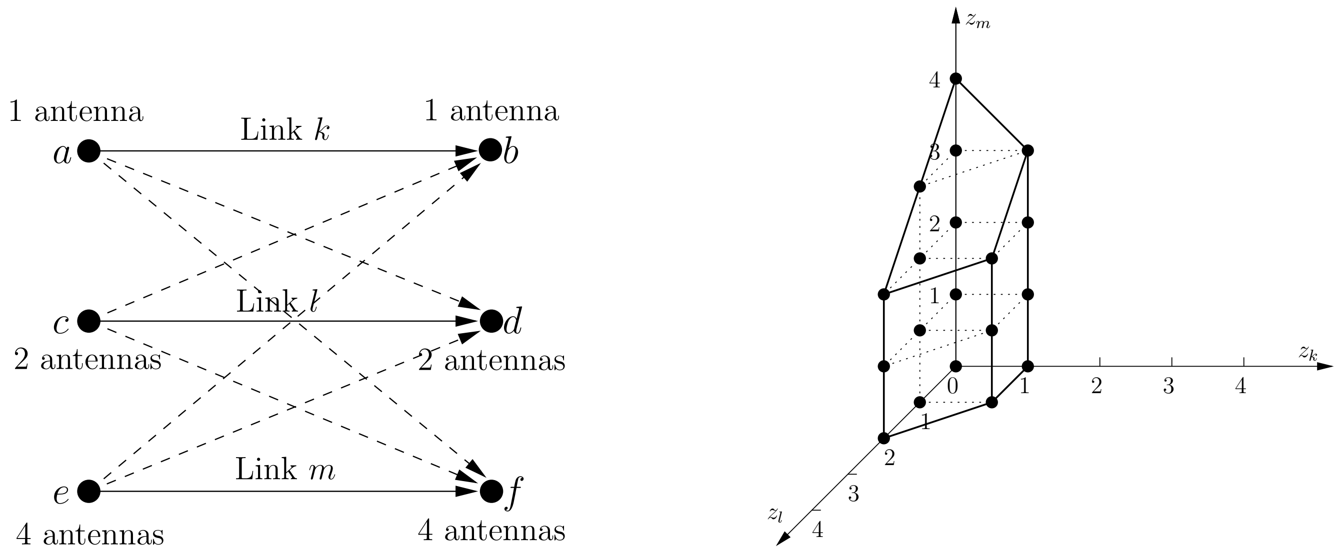
2. DoF constraint at $\text{Rx}(l)$: The number of DoFs that receiver $\text{Rx}(l)$ can use for spatial multiplexing (for reception) and interference cancellation cannot exceed the total number of DoFs at node $\text{Rx}(l)$, i.e.,

$$z_l + \sum_{k \in \mathcal{Q}_l}^{\Pi(\text{Rx}(l)) > \Pi(\text{Tx}(k))} z_k \leq \alpha \,. \tag{5}$$

We use the following simple example to illustrate DoF allocation in a MIMO network.

Example 2 Consider the three-link (k, l, and m) example in Fig. 6(a). The number of antennas at each node is also shown in the figure. Under the above MIMO model, we need an order to determine the DoF resource usage at each node. Suppose we are following an order list, say $a \rightarrow d \rightarrow b \rightarrow c \rightarrow e \rightarrow f$ among the nodes. Then, the DoF allocation in this MIMO network works as follows.

We start with node a, which is the first node in the list. Given it is the first in the list, node a does not have any interference with which it needs to be concerned. Since node a has only 1 antenna, it can transmit at most 1 data stream to its intended receiver b. The second node on the ordered list is node d. Since it appears in the order list after node a, node d needs to suppress the interference from a. This implies that node d needs to expend 1 DoF to cancel the interference from a. Since d has 2 antennas, we have that d can receive at most $2 - 1 = 1$ stream, i.e., $z_l \leq 1$. The DoF consumption on nodes b and c follows exactly the same token, and it can be verified that b and c can each receive and transmit 1 stream, respectively. Since node e's transmission should not interfere with the reception at b and d that had appeared in the order list earlier, e needs to expend 2 DoFs for this purpose. At this point, e can transmit up to $4 - 1 - 1 = 2$ streams, i.e., $z_m \leq 2$. Finally, along the same line, node f can receive at



(a) Inter-nodal interference relationship for three links.

(b) Achievable DoF region of the three links.

Figure 6: A three-link network example.

most $4 - 1 - 1 = 2$ streams, i.e., $z_m \le 2$. Therefore, after the above steps, we can see that the stream combination $(z_k = 1, z_l = 1, z_m = 2)$ can be scheduled feasibly on links k, l, and m. It can be shown that the entire DoF region (the set of all feasible stream combinations) for the three-link example in Fig. 6(a) can be found by enumerating all possible choices of the node order list. Each stream combination offers a feasible point (e.g., $(1, 1, 2)$), the union of which constitutes the DoF region, which we plot in Fig. 6(b). □

6.2 Calculating $f_{\text{RX}}(n)$

Based on the MIMO network model, we now calculate $f_{\text{RX}}(n)$.[4] Recall that $f_{\text{RX}}(n)$ is an upper bound of the maximum number of successful transmissions whose receivers are in the same interference square. In the case of MIMO, this corresponds to the maximum number of successful data streams on all active links whose receivers are in the same interference square.

Lemma 5 *For a random MIMO ad hoc network, we have* $f_{\text{RX}}(n) = \alpha$.

Proof Denote $\mathcal{L}$ the set of active links with their receivers being in the same interference square. Denote $|\mathcal{L}|$ the number of links in $\mathcal{L}$, and let $\mathcal{L} = \{1, \ldots, |\mathcal{L}|\}$. Our goal is to find an upper bound for the sum of data streams on these links, i.e., $\sum_{k \in \mathcal{L}} z_k$.

If $|\mathcal{L}| = 1$, i.e., only one active link with its receiver in the interference square, then $z_1 \le \alpha$ (since the number of data streams on this link cannot exceed the number DoFs of a node). We can set $f_{\text{RX}}(n) = \alpha$ and the lemma holds

[4]For MIMO, the level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as $f_{\text{TX}}(n)$ and either approach will yield the same result.

trivially.

For the general scenario when $|\mathcal{L}| \geq 2$, Property 1 says that these $|\mathcal{L}|$ links interfere with each other and interference cancellation is necessary. Based on the MIMO model we discussed earlier, we need to follow an ordered list for the nodes (both transmitters and receivers) on these $|\mathcal{L}|$ links for DoF allocation at each node. We have two cases, depending on whether the last node in the list is a transmitter or a receiver.

Case (i). The last node in the ordered list is a receiver. Without loss of generality, denote m as the link of which this node is the receiver. To have z_m data streams on link m, based on (5), we have the following constraint on receiver $\text{Rx}(m)$.

$$z_m + \sum_{k \in \mathcal{Q}_m}^{\Pi(\text{Rx}(m)) > \Pi(\text{Tx}(k))} z_k \leq \alpha \ , \tag{6}$$

where the sum for z_k is taken over all interfering links whose transmitters are before receiver $\text{Rx}(m)$ in the node list. Since link m is being interfered by all other links in $\mathcal{L}$ in the same interference square, we have $\mathcal{Q}_m = \mathcal{L} \backslash \{m\}$. Further, since $\text{Rx}(m)$ is the last node in this list, we have $\Pi(\text{Rx}(m)) > \Pi(\text{Tx}(k))$, for all $k \in \mathcal{L} \backslash \{m\}$. Therefore, (6) can be re-written as

$$z_m + \sum_{k \in \mathcal{L} \backslash \{m\}} z_k \leq \alpha \ ,$$

which is

$$\sum_{k \in \mathcal{L}} z_k \leq \alpha \ .$$

Thus, we have shown that the sum of data streams that can be received by nodes in the interference square over all links is upper bounded by α, i.e., $f_{\text{RX}}(n) = \alpha$.

Case (ii). The last node in the ordered list is a transmitter. In this case, we employ (4) and follow the same token as the above discussion. We again have $f_{\text{RX}}(n) = \alpha$.

Combining the two cases, we conclude that $f_{\text{RX}}(n) = \alpha$. □

6.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = \alpha$, we can now apply Theorem 1 and obtain capacity scaling law of a random MIMO ad hoc network as follows.

Proposition 3 *For a random MIMO ad hoc network, we have* $\lambda(n) = O\left(\frac{1}{\sqrt{n \ln n}}\right)$ *almost surely when* $n \to \infty$.

This result is the same as that in [7]. It is also interesting to see that, despite MIMO's ability to increase capacity in a finite-sized network, the scaling order for its asymptotic capacity remains the same as that for the classical single omnidirectional antenna network as in [4].

7 Case Study III: Multi-Channel and Multi-Radio

7.1 Multi-Channel Multi-Radio Model

Multi-channel multi-radio (MC-MR) refers that there are multiple channels in the network and there are multiple radio interfaces at each node in the network [8, 9]. By equipping each node with multiple radio interfaces, each node has more flexibility in accessing the multiple channels in the network. Following [8], we assume that there are c channels in the network and each node in the network is equipped with m radio interfaces, where c and m are constants, and $1 \leq m \leq c$. A radio interface is capable of transmitting or receiving data on only one channel at any given time, i.e., half-duplex.

7.2 Calculating $f_{\text{RX}}(n)$

Based on the MC-MR model, we now calculate $f_{\text{RX}}(n)$.[5] Assuming each band has the same bandwidth in the MC-MR network, then $f_{\text{RX}}(n)$ corresponds to the maximum number of successful transmissions over all available channels on all radio interfaces whose receivers are in the same interference square. We have the following lemma.

Lemma 6 *For a random MC-MR network, we have $f_{RX}(n) = c$.*

Proof Let's focus on one channel at a time. Since the links with receivers in the interference square interfere with each other (Property 1), there can be at most one radio at a node receiving on this channel. Summing up all such radios (or successful transmissions) over c channels, we have $f_{\text{RX}}(n) = c$. □

7.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = c$, we can now apply Theorem 1 and obtain capacity scaling law of an MC-MR ad hoc network as follows.

Proposition 4 *For a random MC-MR ad hoc network, we have $\lambda(n) = O\left(\frac{1}{\sqrt{n \ln n}}\right)$ almost surely when $n \to \infty$.*

Note that this result is the same as the result in [8] for the case when $\frac{c}{m} = O(\ln n)$.

8 Case Study IV: Cognitive Radio Ad Hoc Networks

8.1 Cognitive Radio Network Model

Cognitive radio (CR) is another new physical layer technology that enables more efficient utilization of radio spectrum [19]. A CR is able to constantly sense the radio spectrum and explore any available spectrum bands for data

[5] For an MC-MR network, the level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as $f_{\text{TX}}(n)$ and either approach will yield the same result.

communication. Consider a random ad hoc network where each node is equipped with a CR. Consider a specific time instance where each node i senses a set of available frequency bands $\mathcal{B}_i$ that it can use.[6] Note that due to differences in locations, the set of available frequency bands $\mathcal{B}_i$ at a node i may not be identical to that of another node in the network. Denote $\mathcal{B}_{ij} = \mathcal{B}_i \bigcap \mathcal{B}_j$ the set of common bands that are available at both nodes i and j. Then node i can communicate to node j on band m only if $m \in \mathcal{B}_{ij}$.

8.2 Calculating $f_{\text{RX}}(n)$

Based on the CR network model, we now calculate $f_{\text{RX}}(n)$.[7] Assuming each band has the same bandwidth in the CR network, then $f_{\text{RX}}(n)$ corresponds to the maximum number of successful transmissions over all available bands whose receivers are in the same interference square. Denote $M = |\bigcup_{i=1}^{n} \mathcal{B}_i|$, i.e., M is the number of distinct frequency bands in the network. Then we have the following lemma.

Lemma 7 *For a random CR ad hoc network, we have* $f_{\text{RX}}(n) = M$.

Proof Consider one band at a time. Within each band, by Property 1, the links with receivers in the interference square interfere with each other. So the maximum number of active links (or successful transmissions) is at most one. Summing up all active links (or successful transmissions) over M bands, we have $f_{\text{RX}}(n) = M$. □

8.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = M$, we can now apply Theorem 1 and obtain capacity scaling law of a random CR ad hoc network as follows.

Proposition 5 *For a random CR ad hoc network, we have* $\lambda(n) = O\left(\frac{1}{\sqrt{n \ln n}}\right)$ *almost surely when* $n \to \infty$.

This result is consistent to those found in [5, 14]. It is interesting to see that, despite that CR can utilize spectrum bands more efficiently (and thus higher capacity for a finite-sized network), the scaling order of its asymptotic capacity remains the same as that for the classical single omnidirectional antenna network in [4].

9 Case Study V: Ad Hoc Networks with Multi-Packet Reception

Multi-packet reception (MPR) is a conceptual abstraction of a physical layer capability that a receiver can correctly decode multiple packets from different transmitters simultaneously [17]. As described in [12], such capability may be implemented by a variety of advanced physical layer technologies, such as multiuser detection [18], directional

[6] These bands may be those that are currently unused by the primary users.
[7] For a CR network, the level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as $f_{\text{TX}}(n)$ and either approach will yield the same result.

antennas [11, 20], and MIMO. In other words, MPR refers to a reception capability of a node at the physical layer, rather than referring to a specific physical layer technology. In this section, we employ our criteria in Section 4 to explore capacity scaling law of MPR-based ad hoc networks.

9.1 The MPR Model

In the MPR model, a transmitter can transmit packet to only one receiver at a time, but a receiver is capable of receiving multiple packets simultaneously from multiple transmitters within its transmission range. For unintended transmissions whose interference range covers a receiver, the receiver will consider them as interference. Such interference may be cancelled by the receiver. Specifically, in the MPR model, we assume a receiver has finite resource available for multi-packet reception and interference cancellation. Denote β_1 the number of simultaneous packets from intended transmitters whose transmission range covers the receiver and β_2 the number of unintended transmitters that produce interference on the same receiver. We have

$$\beta_1 + \beta_2 \leq \beta \ ,$$

where β is a constant and represents the total available resource at a receiver. For example, if MIMO is employed to implement MPR, then the number of DoFs at a MIMO node may correspond to β.

Note that this MPR model is a generalization of the idealized MPR model in [12] which assumes $\beta_1 \leq \beta = \infty$ and $\beta_2 = 0$, i.e., a receiver can successfully decode arbitrary number of simultaneous packet transmissions and no interference is allowed on the receiver.

9.2 Calculating $f_{\text{RX}}(n)$

We choose to calculate $f_{\text{RX}}(n)$, which is more convenient than calculating $f_{\text{TX}}(n)$. Recall that $f_{\text{RX}}(n)$ is an upper bound of the maximum number of successful transmissions whose receivers are in the same interference square. In the case of MPR ad hoc networks, $f_{\text{RX}}(n)$ corresponds to an upper bound of the maximum number of packets that are successfully received simultaneously by all the receivers in the same interference square. We have the following lemma for $f_{\text{RX}}(n)$.

Lemma 8 *For a random MPR ad hoc network, we have $f_{\text{RX}}(n) = \beta$.*

Proof Denote $\mathcal{L}$ the set of successful links with their receivers residing in the same interference square. By a "successful" link, we mean the receiver of this link can successfully decode the packet on this link. Denote $|\mathcal{L}|$ the number of links in $\mathcal{L}$, and let $\mathcal{L} = \{1, \ldots, |\mathcal{L}|\}$. Then $f_{\text{RX}}(n)$ is an upper bound of $|\mathcal{L}|$.

Note that for two successful links, their transmitters are different but their receivers may be the same. Consider one receiver j in the interference square. From receiver j's perspective, we divide $\mathcal{L}$ into two subsets: $\mathcal{L}_1$ — the set

Table 2: Summary of capacity scaling laws obtained via our simple criteria for different physical layer technologies. "—" sign indicates new result not yet available in the literature.

Physical Layer Technology		$f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$	Upper Bound	Reference
Directional antenna	Single beam	$f_{\text{TX}}(n) = \Theta(nr^2(n))$	$O(r(n))$	[11]
	Multi-beam	$f_{\text{RX}}(n) = O(n^2r^4(n))$	$O(nr^3(n))$	[11]
MIMO		$f_{\text{RX}}(n) = \alpha$	$O(1/\sqrt{n \ln n})$	[7]
MC-MR		$f_{\text{RX}}(n) = c$	$O(1/\sqrt{n \ln n})$	[8]
CR		$f_{\text{RX}}(n) = M$	$O(1/\sqrt{n \ln n})$	[5, 14]
MPR	Idealized	$f_{\text{RX}}(n) = \Theta(nr^2(n))$	$O(r(n))$	[12]
	General	$f_{\text{RX}}(n) = \beta$	$O(1/\sqrt{n \ln n})$	—

of links whose receivers are j, and $\mathcal{L}_2$ — the set of links whose receivers are not j. Based on Property 1, we know that the transmitters of the links in subset $\mathcal{L}_2$ are all in the interference range of receiver j. Since packets on $\mathcal{L}_1$ are successfully received by j, then based on the MPR model, we have

$$|\mathcal{L}| = |\mathcal{L}_1| + |\mathcal{L}_2| = \beta_1 + \beta_2 \le \beta \ .$$

Therefore, we have $f_{\text{RX}}(n) = \beta$. □

9.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = \beta$, we can now apply Theorem 1 and directly obtain the following capacity scaling law for an MPR-based ad hoc network.

Proposition 6 *For a random MPR ad hoc network, we have* $\lambda(n) = O\left(1/\sqrt{n \ln n}\right)$ *almost surely when* $n \to \infty$.

Remark 1 For the idealized MPR model described in [12], where $\beta_1 \le \beta = \infty$ and $\beta_2 = 0$, one can still apply our simple scaling order criteria. In particular, it can be shown that for this idealized MPR model, $f_{\text{RX}}(n) = \Theta(nr^2(n))$ (see the appendix for details). By Theorem 1, we have $\lambda(n) = O\left(\frac{f_{\text{RX}}(n)}{nr(n)}\right) = O\left(nr^2(n) \cdot \frac{1}{nr(n)}\right) = O\left(r(n)\right)$. This is exactly the result developed in [12]. □

10 Discussions

Summary of Results. Table 2 summarizes capacity scaling laws (upper bounds) that we explored in Sections 5 to 9 by applying our simple scaling order criteria. These upper bounds are the same as those studied in previous work (last column in Table 2), which were developed by various custom-designed approaches. For the MPR general model, there is no prior result available in the literature.

Limitation. Although Table 2 demonstrates the potential capability of our simple scaling order criteria, we caution that the success of our simple criteria hinges upon our successful calculation of $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$. For other physical layer technologies, there is no guarantee that one can always calculate $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$ as we have done in this paper. Further, one needs to calculate $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$ as tight as possible since loose $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$ (e.g., infinity) will yield trivial upper bounds. But one thing that we can guarantee is that should one be able to find $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$ for the underlying physical layer technology, then she can easily apply our simple scaling order criteria to quickly obtain asymptotic upper bound.

Lower Bounds. Note that so far the simple scaling order criteria that we developed in Section 4 can only offer asymptotic capacity upper bounds for different physical layer technologies. A natural question to ask is whether one can develop a set of simple criteria to quickly obtain asymptotic capacity lower bounds for any physical layer technologies. Our efforts to this question have not been fruitful. The main difficulty in deriving a capacity lower bound for a specific physical layer technology is to find a *feasible* solution, which includes resource allocation at physical layer, scheduling at MAC layer, and routing at network layer. A feasible solution to variables at all these layers is much harder to obtain than just developing inequality relationships that are needed to derive asymptotic upper bounds. Given such feasible solution is hard to obtain, whether or not it is possible to develop a unifying approach that yields a set of simple criteria for asymptotic capacity lower bounds remains an open problem.

Despite the absence of a simple criteria for the lower bounds, we may use $\Omega(1/\sqrt{n \ln n})$ (capacity lower bound for single omnidirectional antenna ad hoc networks by G&K [4]) as a lower bound in many cases. This is because single omnidirectional antenna can usually be considered as a special case of these advanced physical layer technologies. In particular, for MIMO, MC-MR, CR, MPR general model in Table 2, we have lower bounds of $\Omega(1/\sqrt{n \ln n})$ and upper bounds of $O(1/\sqrt{n \ln n})$. In these cases, since the upper bound and lower bound have the same scaling order, we conclude that $\lambda(n) = \Theta(1/\sqrt{n \ln n})$ for these advanced physical layer technologies. In other cases where $\Omega(1/\sqrt{n \ln n})$ may appear loose (e.g., single beam and multi-beam directional antenna, idealized MPR), one would need to develop a tighter lower bound by exploiting the unique properties of the underlying physical layer technology.

11 Conclusions

In this paper, we presented a set of simple yet powerful general criteria that one can easily apply to quickly determine the capacity scaling laws for ad hoc networks under the protocol model for various physical layer technologies. Such approach offers a unifying methodology to determine capacity scaling law, which is in contrast to traditional custom-designed approaches. We proved the correctness of our proposed criteria and demonstrate their usage through a number of case studies, such as ad hoc networks with directional antenna, MIMO, MC-MR, cognitive radio, and multiple packet reception. These simple criteria offer a set of powerful tools to networking researchers to understand throughput scaling behavior of ad hoc networks under different physical layer technologies, particularly new technologies that will appear in the future.

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Appendix

Proof of Theorem 1 Recall that we divide the unit square into small interference squares with each having a side length of $1/\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil$ (see Fig. 2). Denote $f_{\text{RX}}(n)$ an upper bound of the maximum number of successful transmissions whose receivers are in the same interference square. Then, the total data rate that each interference square can support is at most $f_{\text{RX}}(n)W$. Now, we can compute the maximum data rate that can be supported by the network in the unit square by taking the sum of the data rates among all small interference squares. Since the side length of each small interference square is $1/\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil$, the total number of small interference squares in the unit area is $\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil^2$. So the maximum data rate that can be supported in the network is at most $\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil^2 f_{\text{RX}}(n)W$.

Let D be the average distance between a source node and its destination node. Since multi-hop routing is employed, we have that the average number of hops for each source-destination pair is at least $\frac{D}{r(n)}$. Note that there are n source-destination pairs. Thus, the required transmission rate over the entire network is at least $\frac{D}{r(n)}n\lambda(n)$.

Since the maximum data transmission that can be supported in the network at a time is $\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil^2 f_{\text{RX}}(n)W$, we have $\frac{D}{r(n)}n\lambda(n) \leq \lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil^2 f_{\text{RX}}(n)W < (\frac{\sqrt{2}}{\Delta\cdot r(n)}+1)^2 f_{\text{RX}}(n)W$, which gives us

$$\lambda(n) < \frac{2f_{\text{RX}}(n)W}{\Delta^2 Dnr(n)} + \frac{2\sqrt{2}f_{\text{RX}}(n)W}{\Delta Dn} + \frac{f_{\text{RX}}(n)Wr(n)}{Dn} = O\left(\frac{f_{\text{RX}}(n)}{nr(n)}\right) . \tag{7}$$

This proves the first part of Theorem 1.

Now, we show the special case when $f_{\text{RX}}(n)$ is a constant. In this case, based on (7), we have

$$\lambda(n) = O\left(\frac{1}{nr(n)}\right) . \tag{8}$$

Note that based on (1), we have $r(n) \geq \sqrt{\frac{\ln n}{n}}$. By substituting $r(n) = \sqrt{\frac{\ln n}{n}}$ into (8), we have

$$\lambda(n) = O\left(\frac{1}{n\sqrt{\frac{\ln n}{n}}}\right) = O\left(\frac{1}{\sqrt{n\ln n}}\right).$$

□

Calculating $f_{\text{RX}}(n)$ for idealized MPR model. We obtain $f_{\text{RX}}(n)$ as follows.

Lemma 9 *For a random ad hoc network under the idealized MPR model, we have $f_{RX}(n) = \Theta(nr^2(n))$.*

Proof First, we show that there can be only one receiver (say j) in the interference square receiving packets. This can be shown by contradiction. Suppose there is another receiver i, $i \neq j$, in the same interference square receiving packets. Then, based on Property 1, a transmitter to receiver i is within the interference range of node j. This transmitter of receiver i will bring interference at node j, which contradicts with $\beta_2 = 0$ under the idealized MPR model.

Although there is only one receiver j receiving packets, it may receive packets from multiple transmitters. Note that all nodes that can transmit to receiver j must fall within the larger square with a side length of $1/\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil+2r(n)$ (see Fig. 5). Based on Lemma 3, we know that the number of all nodes inside the larger square is $\Theta(nr^2(n))$. Since each transmitter transmits one packet to receiver j at a time, the number of simultaneous packets received by receiver j cannot exceed the number of nodes in the larger square, i.e., $\Theta(nr^2(n))$. Therefore, we have $f_{\text{RX}}(n) = \Theta(nr^2(n))$.
□