A General Method to Determine Asymptotic Capacity Upper Bounds for Wireless Networks

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Abstract—Capacity scaling laws offer fundamental understanding on the trend of user throughput behavior when the network size increases. Since the seminal work of Gupta and Kumar, there have been active research efforts in developing capacity scaling laws for ad hoc networks under various advanced physical (PHY) layer technologies. These efforts led to many custom-designed solutions, most of which were mathematically challenging and lacked general properties that can be extended to address scaling laws of ad hoc networks with other PHY layer technologies. So a question is: can we have a general methodology to obtain asymptotic capacity results for various PHY layer technologies? In this paper, we present a simple yet powerful method to determine capacity upper bounds under the protocol model. We prove the correctness of our proposed method and demonstrate its applications to various PHY layer technologies, including directional antenna, MIMO, multi-channel multi-radio, cognitive radio, multiple packet reception, and full-duplex radio. This new method offers a simple tool to researchers to quickly determine asymptotic capacity of wireless networks with a particular PHY layer technology without the need to resort to complex custom-designed analysis as done in the literature.

Index Terms—Asymptotic capacity, upper bounds, scaling law, protocol model, physical layer technology.

1 Introduction

CAPACITY scaling laws refer to how a user's throughput scales as the network size increases to infinity.[1] Such scaling law results, expressed in $O(\cdot)$, $\Omega(\cdot)$, and $\Theta(\cdot)$ as a function of n (where n is the number of nodes in the network and approaches infinity), offer fundamental understanding on the trend of user throughput behavior when the network size increases.

Since the seminal results of Gupta and Kumar ("G&K" for short) on capacity scaling law of ad hoc networks with single omnidirectional antennas [7], there has been a growing body of research efforts on exploring capacity scaling laws for ad hoc networks under various physical (PHY) layer technologies. These include directional antenna [15], [25], MIMO [10], multi-channel multi-radio (MC-MR) [12], cognitive radios [8], [9], [18], [26], multiple packet reception (MPR) [16], and full-duplex [24], among others. For each of these advanced PHY layer technologies, a *custom-designed* analytical approach was developed to study its capacity scaling law. Most of these solutions were mathematically challenging and lacked general properties that can be extended to address scaling laws of wireless networks with other PHY layer technologies.

A fundamental question we ask in this paper is the following. Instead of custom-designing an analysis for each PHY layer technology, can we devise a set of simple yet general method that can be easily and quickly applied to determine capacity scaling laws for various PHY layer technologies? If successful, this new method will serve as a powerful tool to networking researchers to study and understand throughput scaling behavior of wireless networks under various PHY layer technologies, both current and future.

The main contribution of this paper is the development of a simple method for establishing capacity upper bounds under the protocol model for wireless networks under various PHY layer technologies. The following is a summary of our contributions.

- We give an in-depth study of G&K's analysis on asymptotic capacity bound for ad hoc networks with single omnidirectional antennas. We offer insight on why their approach cannot be applied to analyze asymptotic capacity under some other PHY layer technologies.
- We propose a new and novel method based on the so-called "interference square" concept. Under this concept, we divide a normalized 1×1 network area into small interference squares, each with side length $1/\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \rceil$, where $r(n)$ is the transmission range and Δ is a parameter to set the interference range under the protocol model. For transmissions within an interference square, we show some unique interference properties.
- Based on the new interference square concept, we develop two simple yet powerful scaling order criteria to determine the asymptotic capacity upper bounds for various PHY layer technologies. Either criterion is sufficient to give a capacity upper bound for a given PHY layer technology, and the choice of which criterion to use is purely a matter of convenience depending on the underlying problem. We also prove the correctness of applying these criteria in obtaining capacity upper bounds.
- To demonstrate the application of our proposed method, we study asymptotic capacity of wireless networks under

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1. When there is no ambiguity, we use the terms "asymptotic capacity" and "capacity scaling law" interchangeably throughout this paper.

various PHY layer technologies, including directional antenna, MIMO, MC-MR, cognitive radio, MPR, and full-duplex. We show that by applying our simple method, one can easily obtain capacity upper bounds under these PHY layer technologies. This is in sharp contrast to similar results developed in the literature, which involved complex mathematical analysis that was custom-designed for each PHY layer technology. Note that our method not only can quickly validate those results already reported in the literature, it can also quickly determine some new results that have not been studied before. Further, it can be a useful tool to study wireless networks under other new PHY layer technologies in the future.

Just like any useful tool, our proposed method is not without limitations and several disclaimers are in order. First, our method is developed to determine capacity upper bound. It should not be too surprising that there does not appear to exist a general method to determine lower bound. This is because finding a capacity lower bound requires to find a good and *feasible* solution, which must be tied to the specific underlying PHY layer technology. Typically, a feasible solution includes resource allocation at the physical layer, scheduling at the MAC layer, and routing at the network layer, each of which is dictated by the underlying PHY technology. This is in contrast to the development of asymptotic upper bounds, for which one can exploit inequality relationships (rather than ensuring absolute feasibility). Second, as we explicitly stated in the paper title, our method is developed solely under the protocol model. Developing a unified method under the SINR-based interference model remains an open problem. This limitation is partially due to the fact that it remains unknown whether there exists a general SINR-based model for different PHY layer technologies. More discussion on this is given in our conclusions at the end of the paper (Section 12). Third, we have only considered the wireless network scenario where nodes are uniformly distributed in an area. Although some works considered non-uniform node distribution [1], [2], it remains an open problem whether our approach can be extended to such cases (with non-uniform node distribution).

The remainder of this paper is organized as follows. In Section 2, we take a closer look at G&K's classical method (for wireless networks with single omnidirectional antennas) and understand why it cannot serve as a general method to analyze other PHY layer technologies. Subsequently, in Section 3, we propose a novel interference square concept and based on this concept, in Section 4, we present two simple yet powerful scaling order criteria, which can be used to easily and quickly derive capacity upper bounds for various PHY layer technologies. We also give a simple benchmark for the lower bounds in the absence of a general method to find asymptotic lower bounds. As applications of our proposed method, in Sections 5 to 10, we apply it to wireless networks based on various PHY layer technologies such as directional antenna, MIMO, MC-MR, cognitive radio, MPR, and full-duplex. Section 11 offers discussions of our work. Section 12 concludes this paper. Table 1 lists notation used in this paper.

TABLE 1
Notation.

General notation	
d_{ij}	Distance between nodes i and j
D	Average distance between all source-destination pairs
$f_{\text{RX}}(n)$	An upper bound for the maximum number of successful transmissions whose receivers are in the same interference square
$f_{\text{TX}}(n)$	An upper bound for the maximum number of successful transmissions whose transmitters are in the same interference square
n	The number of nodes in the network
$\mathcal{N}$	The set of nodes in the network
W	The data rate of a successful transmission in a channel
$r(n)$	The (common) transmission range of all nodes under the protocol model
$\text{Rx}(l)$	Receiver of link l
$\text{Tx}(l)$	Transmitter of link l
Δ	A parameter to set interference range in the protocol model
$\lambda(n)$	Per-node throughput of a random network with n nodes
Ad hoc network with directional antennas	
S	An interference square in the unit area
A_S	Area of S
N_S	Number of nodes in S
MIMO ad hoc network	
$\mathcal{I}_l$	The set of links that are interfered by link l
$\mathcal{Q}_l$	The set of links that are interfering link l
z_l	Number of data streams on link l
α	Number of antennas at each node
$\Pi(\cdot)$	The mapping between a node and its order in the node list
MC-MR network	
c	The number of channels in the network
m	The number of radio interfaces at each node
CR ad hoc network	
$\mathcal{B}_i$	The set of available bands at node i
$\mathcal{B}_{ij}$	The set of available bands on link (i, j)
M	$= \lvert \bigcup_{i=1}^{n} \mathcal{B}_i \rvert$, i.e., the number of distinct frequency bands in the network
Ad hoc network with MPR	
β_1	Number of simultaneous packets from intended transmitters whose transmission range covers a receiver
β_2	Number of unintended transmitters that produce interference on the same receiver
β	A constant representing the total available resource at a receiver

2 Lesson Learned From G&K's Classical Approach

In this section, we take a close look at G&K's classical approach in analyzing capacity scaling law and try to understand why such an approach becomes a barrier in analyzing capacity scaling laws when other PHY layer technologies are employed.

2.1 Background

In G&K's work [7], they considered an ad hoc network of n nodes that are randomly located within a unit square area. Each node in the network is a source node and transmits its data to a randomly chosen destination node. A node's transmission is limited by its transmission range. When the distance between a source node and its destination node is large, multi-hop routing is needed to relay the data. The per-node throughput $\lambda(n)$ is defined as the data rate that can be sent from each source to its destination. A capacity scaling law attempts to characterize the maximum per-node throughput $\lambda(n)$ when the number of nodes n goes to infinity.

In [7], two interference models, the protocol model and the physical model, were considered in their study. In the protocol model [7], each transmitting node is associated with a transmission range $r(n)$, and an interference range $(1 + \Delta)r(n)$, where Δ is a constant. To guarantee the connectivity of the network, transmission range $r(n)$ must satisfy the following condition (regardless

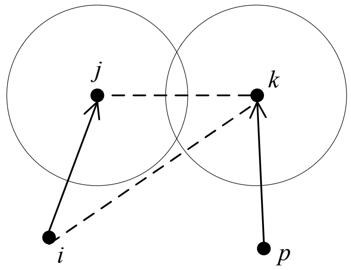


Fig. 1. Overlapping of two circular footprints of two receiving nodes.

of the underlying physical layer technology) [6]:

$$r(n) \geq \sqrt{\frac{\ln n}{n}} \ . \quad (1)$$

When node i transmits to node j, the necessary and sufficient conditions for a successful transmission are:

- node j is within the transmission range of node i, i.e., $d_{ij} \leq r(n)$, where d_{ij} is the distance between nodes i and j, and
- for any transmitting node k other than node i, node j is outside the interference range of node k, i.e., $d_{kj} > (1+\Delta)r(n)$, if k is a transmitting node and $k \neq i$.

In [7], when the transmission from a node to another node is successful, then the achieved data rate for this transmission is assumed to be a constant W.

2.2 G&K's Approach and Its Limitation

A key component in G&K's approach (in deriving capacity upper bound) is to calculate how much footprint area each successful transmission occupies. Then by dividing the unit square area by this area, they were able to obtain an upper bound of the maximum number of successful transmissions at a time and subsequently to derive a capacity upper bound. Specifically, in [7], G&K showed that for a successful reception at each receiver, one can draw a circle around each receiver with radius $\frac{\Delta r(n)}{2}$ and these circles must be disjoint.[2] Under the above approach, a successful transmission will occupy a circular footprint area of at least $\pi\left[\frac{\Delta r(n)}{2}\right]^2$. Then the maximum number of successful transmissions within the unit square area is at most $1/\left\lfloor \pi(\frac{\Delta r(n)}{2})^2 \right\rfloor$ at any time. Based on this result, G&K derived a capacity upper bound.

The essence of the above footprint area approach is to identify the size of the circular area that each successful transmission will occupy. But this approach poses a barrier when we encounter other PHY layer technologies (e.g., MIMO, directional antenna) beyond single omnidirectional antenna node considered in [7]. This is because under these advanced PHY layer technologies, the interference relationships among the nodes are much more complex than those under the single omnidirectional antenna scenario in [7]. In particular, the footprint area of each successful

2. This result can be proved by contradiction. That is, suppose two circles centered at receivers j and k with radius $\frac{\Delta r(n)}{2}$ are not disjoint (see Fig. 1), then $d_{jk} \leq \Delta r(n)$. Suppose receiver j is receiving data from transmitter i. Then we have $d_{ij} \leq r(n)$. Based on the triangle inequality, we have $d_{ik} \leq d_{ij} + d_{jk} \leq (1+\Delta)r(n)$, which means that receiver k is within the interference range of i. But this contradicts with the fact that receiving node k must fall outside of the interference range of node i.

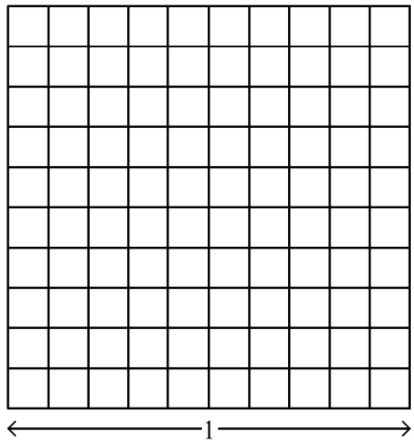


Fig. 2. The unit square is divided into equal-sized small interference squares, each with a side length of $1/\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \rceil$.

receiver does *not* have to be disjoint. For example, in a MIMO ad hoc network where each node employs multiple transmit/receive antennas, receiving node k in Fig. 1 may use its degree-of-freedoms (DoFs) to cancel the interference from transmitting node i [3], [19]. As a result, G&K's approach of associating disjoint footprint area with each successful transmission falls apart.

3 A New Approach

Given that the footprint area approach in [7] is not capable of handling more complex interference relationships (brought by other PHY layer technologies), we propose a new approach that handles interference from a different perspective.

We consider the same network setting as in G&K's work [7], where there is an ad hoc network of n nodes that are randomly located within a unit square area. Each node in the network is a source node and transmits its data to a randomly chosen destination node. A node's transmission is limited by its transmission range. When the distance between a source node and its destination node is large, multi-hop routing is needed to relay the data.

In our new approach, instead of focusing on how much footprint area each successful transmission occupies, we will calculate how many successful transmissions that a given small area in the network can support. Specifically, we divide the unit square into small equal-sized squares (Fig. 2), each with a side length of $1/\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \rceil$. We call each small square an *interference square.* As we shall show in Section 4, if one can find the maximum number of successful transmissions in each interference square (under a specific PHY layer technology), then we can derive the capacity upper bound for the entire network. Subsequently, in Sections 5 to 9, we show how to find the maximum number of successful transmissions in each interference square under different PHY layer technologies, thus deriving capacity upper bound for each of these technologies.

Before we show how this new interference square approach can offer simple scaling law criteria, we discuss some important properties associated with a small square as follows.

Property 1. For a set of successful simultaneous transmissions whose receivers fall in the same interference square, the

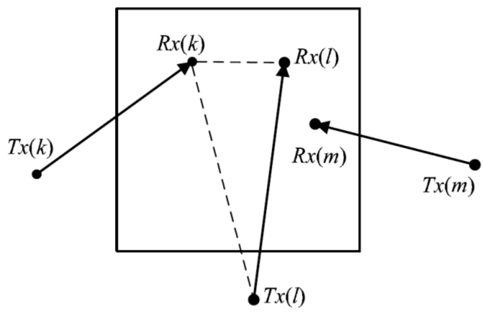


Fig. 3. A set of transmissions whose receivers are in the same interference square.

receiver of any such transmission must be within the interference range of any other transmitter from the same set of transmissions.

Proof: Note that the distance between any two receivers in the same interference square is at most $\sqrt{2}\cdot 1/\left[\frac{\sqrt{2}}{\Delta\cdot r(n)}\right] = \sqrt{2}\cdot\frac{\Delta r(n)}{\sqrt{2}} = \Delta\cdot r(n)$. Denote $\mathrm{Tx}(l)$ and $\mathrm{Rx}(l)$ the transmitter and receiver of transmission l, respectively. Referring to Fig. 3, for any two transmissions l and k with their receivers $\mathrm{Rx}(l)$ and $\mathrm{Rx}(k)$ in the interference square, we have $d_{\mathrm{Rx}(l),\mathrm{Rx}(k)} \le \Delta\cdot r(n)$. Since $d_{\mathrm{Tx}(l),\mathrm{Rx}(l)} \le r(n)$ (recall that $r(n)$ is transmission range) based on the triangle inequality, we have $d_{\mathrm{Tx}(l),\mathrm{Rx}(k)} \le d_{\mathrm{Rx}(l),\mathrm{Rx}(k)} + d_{\mathrm{Tx}(l),\mathrm{Rx}(l)} \le (1+\Delta)r(n)$. Similarly, we can prove that the receiver $\mathrm{Rx}(l)$ of transmission l is also in the interference range of transmitter $\mathrm{Tx}(k)$ of transmission k. □

Similar to Property 1 (which considers receivers in the same interference square), we can consider transmitters in the same interference square and have the following property.

Property 2. For a set of successful simultaneous transmissions whose transmitters reside in the same interference square, the receiver of any such transmission must be within the interference range of any other transmitter from the same set of transmissions.

The proof of Property 2 is similar to that of Property 1 and is omitted.

Properties 1 and 2 show us two complementary ways to assess interference relationship from either receiver or transmitter perspective in the same interference square. It turns out that these two properties allow us to calculate the number of successful transmissions with either their receivers or transmitters in the same interference square under various PHY layer technologies. For example, under the single omnidirectional antenna setting in Section 2.1, we can easily conclude that there can be at most one active receiver (or transmitter) in an interference square for a successful transmission, i.e., the maximum number of successful transmissions with either receivers or transmitters in the same interference square is one. As another example, for MIMO ad hoc network where each node is equipped with multiple transmit/receiver antennas, Properties 1 and 2 allow us to show that the maximum number of successful transmissions whose receivers (or transmitters) in the same interference square is upper bounded by the number of antennas at each node (see details in Section 6). As we shall show in the next section (Theorems 1 and 2), the maximum number of successful transmissions whose receivers (or transmitters) are in the same interference square will determine the capacity scaling law of an ad hoc network under various PHY layer technologies.

4 Main Results: Simple Scaling Order Criteria

As we shall show in Sections 5 to 10, for a specific PHY layer technology, the newly defined interference square and Properties 1 and 2 enable us to characterize the maximum number of successful transmissions whose receivers (or transmitters) are in the same interference square. For a specific PHY layer technology, denote

- $f_{\mathrm{RX}}(n)$ as an upper bound for the maximum number of successful transmissions whose *receivers* are in the same interference square.

Similarly, denote

- $f_{\mathrm{TX}}(n)$ as an upper bound for the maximum number of successful transmissions whose *transmitters* are in the same interference square.

In this section, we show that once we have either $f_{\mathrm{RX}}(n)$ or $f_{\mathrm{TX}}(n)$, we can quickly determine a capacity scaling order. Figure 4 summarizes the idea of the above discussion.

The two criteria that we present in this section (Theorem 1 and 2) show that the capacity upper bound scales asymptotically with either $\frac{f_{\mathrm{RX}}(n)}{nr(n)}$ or $\frac{f_{\mathrm{TX}}(n)}{nr(n)}$. We formally state these results as follows.

Theorem 1 (Criterion 1). For a given $f_{\mathrm{RX}}(n)$, the asymptotic capacity upper bound of a random ad hoc network is $\lambda(n) = O\left(\frac{f_{\mathrm{RX}}(n)}{nr(n)}\right)$ almost surely when $n\to\infty$. In the special case when $f_{\mathrm{RX}}(n)$ is a constant, then $\lambda(n) = O(1/\sqrt{n\ln n})$ almost surely when $n\to\infty$.

Proof: Recall that we divide the unit square into small interference squares with each having a side length of $1/\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil$ (see Fig. 2). Denote $f_{\mathrm{RX}}(n)$ an upper bound of the maximum number of successful transmissions whose receivers are in the same interference square. Then, the total data rate that each interference square can support is at most $f_{\mathrm{RX}}(n)W$. Now, we can compute the maximum data rate that can be supported by the network in the unit square by taking the sum of the data rates among all small interference squares. Since the side length of each small interference square is $1/\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil$, the total number of small interference squares in the unit area is $\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil^2$. So the maximum data rate that can be supported in the network is at most $\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil^2 f_{\mathrm{RX}}(n)W$.

Let D be the average distance between a source node and its destination node. Since multi-hop routing is employed, we have that the average number of hops for each source-destination pair is at least $\frac{D}{r(n)}$. Note that there are n source-destination pairs. Thus, the required transmission rate over the entire network is at least $\frac{D}{r(n)}n\lambda(n)$.

Since the maximum data transmission that can be supported in the network at a time is $\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\rceil^2 f_{\mathrm{RX}}(n)W$, we have $\frac{D}{r(n)}n\lambda(n) \le \left\lceil\frac{\sqrt{2}}{\Delta\cdot r(n)}\right\rceil^2 f_{\mathrm{RX}}(n)W < \left(\frac{\sqrt{2}}{\Delta\cdot r(n)}+1\right)^2 f_{\mathrm{RX}}(n)W$, which gives us

$$\begin{aligned}\lambda(n) &< \frac{2f_{\mathrm{RX}}(n)W}{\Delta^2 Dnr(n)} + \frac{2\sqrt{2}f_{\mathrm{RX}}(n)W}{\Delta Dn} + \frac{f_{\mathrm{RX}}(n)Wr(n)}{Dn} \\ &= O\left(\frac{f_{\mathrm{RX}}(n)}{nr(n)}\right). \qquad (2)\end{aligned}$$

This proves the first part of Theorem 1.

Now, we show the special case when $f_{\text{RX}}(n)$ is a constant. In this case, based on (2), we have

$$\lambda(n) = O\left(\frac{1}{nr(n)}\right) . \quad (3)$$

Note that based on (1), we have $r(n) \geq \sqrt{\frac{\ln n}{n}}$. By substituting $r(n) = \sqrt{\frac{\ln n}{n}}$ into (3), we have $\lambda(n) = O\left(\frac{1}{n\sqrt{\frac{\ln n}{n}}}\right) = O\left(\frac{1}{\sqrt{n \ln n}}\right)$. □

Similarly, if we can find $f_{\text{TX}}(n)$, then the following criterion can also give an upper bound for the asymptotic capacity.

Theorem 2 (Criterion 2). For a given $f_{\text{TX}}(n)$, the asymptotic capacity upper bound of a random ad hoc network is $\lambda(n) = O\left(\frac{f_{\text{TX}}(n)}{nr(n)}\right)$ almost surely when $n \to \infty$. In the special case when $f_{\text{TX}}(n)$ is a constant, then $\lambda(n) = O(1/\sqrt{n \ln n})$ almost surely when $n \to \infty$.

The proof of Theorem 2 is similar to that of Theorem 1 and is omitted to conserve space.

Several remarks about the above two criteria are in order.

- First, for a specific PHY technology, we only need to focus on the calculation of either $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, whichever is more convenient. An asymptotic capacity upper bound will follow once we have either $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, based on either Theorem 1 or Theorem 2.
- Second, when either $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$ is a constant, then the asymptotic capacity upper bound is $O(1/\sqrt{n \ln n})$, which is precisely the same as that in [7] by G&K for the protocol model. This offers a quick test on whether the underlying PHY technology will indeed change the scaling order of the classical single omnidirectional antenna based ad hoc network in [7].
- Finally, the two criteria allow us to focus on calculation ($f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$) only within a small interference square. The details associated with network-wide multi-hop end-to-end throughput have been folded in the proof of the two theorems and are no longer of concerns to users of these two theorems in deriving asymptotic capacity upper bound for a given PHY technology.

Example 1. As the first application of our scaling order criterion, let's validate the single omnidirectional antenna based ad hoc network considered in [7]. As discussed in Section 3, we have that $f_{\text{RX}}(n) = 1$. Thus, by Theorem 1, we have $\lambda(n) = O(1/\sqrt{n \ln n})$, which is precisely the same result in [7] by G&K.

In the remaining several sections, we will explore asymptotic capacity upper bounds for ad hoc networks under various PHY technologies. We will present results for directional antennas, MIMO, MC-MR, cognitive radio, MPR, and full-duplex radio in this paper. Referring to Fig. 4, for each case, we will first calculate either $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, whichever is more convenient, based on the new interference square and Properties 1 and 2. This is the upper righthand block in Fig. 4. Once we have $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$, then we will apply one of the two criteria in this section to quickly obtain the capacity scaling law for this PHY technology (bottom block in Fig. 4).

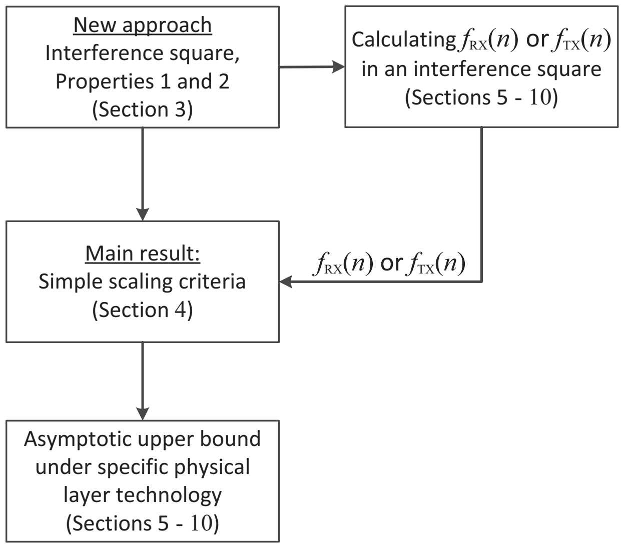


Fig. 4. A flow chart illustrating our approach to derive asymptotic upper bound for a specific physical layer technology.

Recall our earlier discussion that a simple method to obtain capacity lower bounds is not possible due to the need of finding a good and feasible solution, which is closely tied to the specific PHY technology. Nevertheless, we may use $\Omega(1/\sqrt{n \ln n})$ (capacity lower bound for single omnidirectional antenna ad hoc networks by G&K [7]) as a benchmark lower bound in many cases. This is because single omnidirectional antenna can be considered as a special case of some of these advanced PHY technologies. If this crude lower bound has the same scaling order as the upper bound that we find for a particular PHY technology, then we can confidently conclude that $\lambda(n) = \Theta(1/\sqrt{n \ln n})$. Otherwise, $\Omega(1/\sqrt{n \ln n})$ may appear loose, and we would need to develop a tighter lower bound by exploiting the unique properties of the underlying PHY technology. We will experience both cases in the following case studies.

5 Case Study I: Ad Hoc Networks with Directional Antennas

Compared to omnidirectional antenna, directional antenna can control its beam width and concentrate its beam toward its intended destination. Since nodes outside the beam is not interfered, greater spatial reuse inside the network can be achieved. In this section, we apply our method in Section 4 to explore asymptotic capacity of a random ad hoc network with each node being equipped with a directional antenna. We follow the same model as in [15] by Peraki and Servetto.[3] The scaling law results in [15] are well known and widely cited. They showed that for the single-beam model, the asymptotic capacity scales as $O(r(n))$ and for the multi-beam model, it scales as $O\left(nr^3(n)\right)$. The analysis in [15] was custom-designed and differed from that by G&K. The analysis required significant efforts in its construction. In contrast, in this section, we show that by applying our simple method in

3. Another work on scaling law for directional antennas is [25] by Yi *et al.*, which employed a slightly different model and thus led to a different set of results. The approach in [25] followed the same token as that in [7] by G&K. It can be shown that our criteria can be easily applied there and we leave the details to readers as an exercise.

Section 4, we can quickly obtain (using less than 1.5 pages) the same results for asymptotic capacity upper bound in [15]. We organize this section as follows. First, we consider the case for the single-beam model. Then, we consider the multi-beam model.

5.1 Scaling Law Analysis for Single Beam Model

5.1.1 Single Beam Model

The protocol model for single beam model is defined as follows [15].

- A transmitter can generate at most one directional beam to an intended receiver within its transmission range.
- A receiver can receive multiple directional beams from different transmitters where the receiver is within their transmission range, as long as these transmitters do not lie on the same line.

5.1.2 Calculating $f_{TX}(n)$

In this case study, we choose to calculate $f_{\text{TX}}(n)$, which is more convenient than $f_{\text{RX}}(n)$. As discussed in Section 4, the choice of calculating $f_{\text{TX}}(n)$ or $f_{\text{RX}}(n)$ is solely based on convenience and either one is sufficient to determine asymptotic capacity.

Recall that $f_{\text{TX}}(n)$ is an upper bound for the maximum number of successful transmissions whose transmitters are in the same interference square. In the case of single-beam model, $f_{\text{TX}}(n)$ corresponds to an upper bound for the maximum number of successful beam transmissions whose transmitters are in the same interference square. To calculate $f_{\text{TX}}(n)$, we need the following lemma.

Lemma 1. The number of nodes in the same interference square is $\Theta\left(nr^2(n)\right)$ almost surely when $n \to \infty$.

Proof: Denote S as an interference square within the unit area. Denote A_S and N_S the area and the number of nodes in S, respectively. Since nodes in S are randomly distributed, we have the average number of nodes in S is $E(N_S) = nA_S$. For the number of nodes in S, we have the following probabilities (also known as Chernoff bounds) [14].

$$P\left\{N_S > (1+\delta)nA_S\right\} < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{nA_S} \quad \text{for any } \delta > 0 ,$$

$$P\left\{N_S < (1-\delta)nA_S\right\} < e^{-\frac{1}{2}nA_S\delta^2} \quad \text{for any } 0 < \delta < 1 .$$

Combining the above two inequalities, for any $0 < \delta < 1$, we have

$$\begin{aligned} & P\left\{|N_S - nA_S| > \delta nA_S\right\} \\ = \; & P\left\{N_S > (1+\delta)nA_S\right\} + P\left\{N_S < (1-\delta)nA_S\right\} \\ < \; & \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{nA_S} + e^{-\frac{1}{2}nA_S\delta^2} \\ = \; & e^{-\theta_1 nA_S} + e^{-\theta_2 nA_S} , \end{aligned} \tag{4}$$

where $\theta_1 = (1+\delta)\ln(1+\delta) - \delta$ and $\theta_2 = \frac{1}{2}\delta^2$.

Note that $A_S = 1/\left\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \right\rceil^2 = \Theta(r^2(n))$. Letting $A_S = \Theta(r^2(n))$ in (4), we have

$$P\left\{|N_S - nA_S| > \delta nA_S\right\} < e^{-\theta_1 n\Theta(r^2(n))} + e^{-\theta_2 n\Theta(r^2(n))} . \tag{5}$$

Based on (1), we have $r(n) = \Omega(\sqrt{\frac{\ln n}{n}})$. Thus, the right-hand-side of (5) goes to zero when $n \to \infty$, which shows that the probability that the deviation of the number of nodes in S from the mean by more than a constant factor of the mean is zero when $n \to \infty$. Based on the definition of $\Theta(\cdot)$, we have $N_S = \Theta(nr^2(n))$. □

Based on Lemma 1, we have the following lemma for $f_{\text{TX}}(n)$.

Lemma 2. For a random ad hoc network under single-beam directional antenna, we have $f_{\text{TX}}(n) = \Theta\left(nr^2(n)\right)$.

Proof: By Lemma 1, there are $\Theta\left(nr^2(n)\right)$ nodes in the interference square. Since each node can only generate one beam, the total number of successful beam transmissions generated by the transmitters in this interference square cannot exceed $\Theta\left(nr^2(n)\right)$, i.e., $f_{\text{TX}}(n) = \Theta\left(nr^2(n)\right)$. □

5.1.3 Scaling Law

Following Fig. 4, with $f_{\text{TX}}(n) = \Theta\left(nr^2(n)\right)$, we can now apply Theorem 2 and quickly obtain the following asymptotic capacity upper bound.

Proposition 1. For a random ad hoc network under single-beam directional antenna, we have $\lambda(n) = O\left(r(n)\right)$ almost surely when $n \to \infty$.

Proof: Combining Lemma 2 and Theorem 2, we have

$$\lambda(n) = O\left(\frac{f_{\text{TX}}(n)}{nr(n)}\right) = O\left(nr^2(n) \cdot \frac{1}{nr(n)}\right) = O\left(r(n)\right) .$$

□

Note that this result for single-beam case is the same as that in [15]. This upper bound is tight since it has the same asymptotic order as the lower bound obtained in [15].

5.2 Scaling Law Analysis for the Multi-Beam Model

5.2.1 Multi-Beam Model

The protocol model for multi-beam model is defined as follows [15].

- A transmitting node can generate multiple beams to different receiving nodes within its transmission range at the same time.
- A receiving node can only receive one beam from the same transmitting node but may receive multiple beams from different transmitting nodes where the receiver is within their transmission range, as long as these transmitters do not lie on the same straight line.

5.2.2 Calculating $f_{RX}(n)$

We will calculate $f_{\text{RX}}(n)$.[4] Recall that $f_{\text{RX}}(n)$ is an upper bound of the maximum number of successful transmissions whose receivers are in the same interference square. In the case of multi-beam model, $f_{\text{RX}}(n)$ corresponds to an upper bound of the maximum number of successful beam transmissions received by the receivers that are in the same interference square.

For receivers residing in the same interference square, it is easy to see that their transmitters cannot be outside a larger square, with the same center as the interference square, but with side length of $1/\lceil \frac{\sqrt{2}}{\Delta \cdot r(n)} \rceil + 2r(n)$ (see Fig. 5). Otherwise, a receiver in the interference square will be outside of a transmitter's transmission range $r(n)$. For the number of nodes inside the larger square

4. The level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as that for $f_{\text{TX}}(n)$ in the multi-beam model. Either choice will lead to the same result.

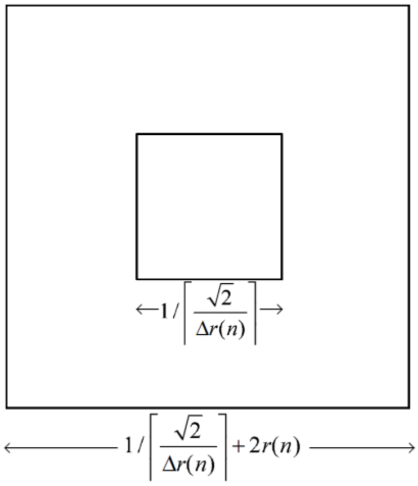


Fig. 5. The larger square contains all the transmitters that can transmit directional beams to the receivers that are in the small interference square at the center.

(regardless of transmitters or receivers), we have the following lemma.

Lemma 3. The number of nodes in the larger square with side length $2r(n)+1/\left\lceil \frac{\sqrt{2}}{\Delta\cdot r(n)} \right\rceil$ is $\Theta\left(nr^2(n)\right)$ almost surely when $n \to \infty$.

The proof of Lemma 3 is similar to the proof of Lemma 1 and is omitted here. Now, we are ready to calculate $f_{\text{RX}}(n)$ as follows.

Lemma 4. For a random ad hoc network under multi-beam directional antenna, we have $f_{\text{RX}}(n) = O\left(n^2 r^4(n)\right)$.

Proof: Based on Lemma 3, we know that the number of transmitters that can transmit beams to the same receiver in the interference square is at most $O\left(nr^2(n)\right)$. That is, a receiver in the interference square can receive at most $O\left(nr^2(n)\right)$ beams. By Lemma 1, there are at most $\Theta\left(nr^2(n)\right)$ receivers in the same interference square. So we have

$$f_{\text{RX}}(n) = \Theta\left(nr^2(n)\right) \cdot O\left(nr^2(n)\right) = O\left(n^2 r^4(n)\right) .$$

□

5.2.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = O\left(n^2 r^4(n)\right)$, we can now apply Theorem 1 and quickly obtain the following asymptotic capacity upper bound.

Proposition 2. For a random ad hoc network under multi-beam directional antenna, we have $\lambda(n) = O\left(nr^3(n)\right)$ almost surely when $n \to \infty$.

Proof: Combining Lemma 4 and Theorem 1, we have

$$\lambda(n) = O\left(\frac{f_{\text{RX}}(n)}{nr(n)}\right) = O\left(n^2 r^4(n) \cdot \frac{1}{nr(n)}\right) = O\left(nr^3(n)\right) .$$

□

This result is the same as that in [15] for the multi-beam case. This upper bound is tight since it has the same asymptotic order as the lower bound obtained in [15].

6 Case Study II: MIMO Ad Hoc Networks

6.1 MIMO Model

By employing multiple antennas at both transmitting and receiving nodes, MIMO has brought significant benefits to wireless communications, such as increased link capacity [4], [20], improved link diversity [28], and interference cancellation between conflicting links [3], [19]. In this section, we characterize asymptotic capacity upper bound for multi-hop MIMO ad hoc networks. Although there are many schemes to exploit the benefits of antenna arrays at a node, we focus on the two key characteristics of MIMO: *spatial multiplexing* (SM) and *interference cancellation* (IC) [3], [19], [27]. SM refers that a transmitter can send several independent data streams to its intended receiver simultaneously on a link. IC refers that by properly exploiting multiple antennas at a node, potential interference to and/or from other nodes can be cancelled.

To model SM and IC, we employ recent advance in MIMO protocol model in [17] by Shi *et al.* The MIMO protocol model is defined as follows. In this model, degree-of-freedom (DoF) is used to represent resource at a MIMO node. Simply put, the number of DoFs at a node is equal to the number of antennas, denoted as α, at the node. Denote z_l the number of active data streams on link l in a time slot. Denote Tx(l) and Rx(l) the transmitter and the receiver of link l, respectively. To spatial multiplex z_l data streams on link l, we need to allocate z_l ($z_l \le \alpha$) DoFs at both transmitter Tx(l) and receiver Rx(l). To cancel interference from and/or to other nodes in the network, it is necessary to have an ordered list for all nodes and allocate DoFs at each node following this order [17]. Denote $\Pi(\cdot)$ the mapping between a node and its order in the node list. Suppose that link l is carrying z_l data streams. Denote $\mathcal{I}_l$ and $\mathcal{Q}_l$ the set of links that are interfered by link l and the set of links that are interfering link l, respectively. Transmitter Tx(l) is responsible for cancelling the interference from itself to all receivers Rx(k), $k \in \mathcal{I}_l$, that are before node Tx(l) in the order list. Similarly, receiver Rx(l) of link l is responsible for cancelling the interference from all transmitters Tx(k), $k \in \mathcal{Q}_l$, that are before node Rx(l) in the order list. Since the total number of DoFs for SM and IC cannot exceed α, we have the following two constraints on each active link l in the network.

1) DoF constraint at Tx(l): The number of DoFs that Tx(l) can use for SM (for transmission) and IC cannot exceed the total number of DoFs at node Tx(l), i.e.,

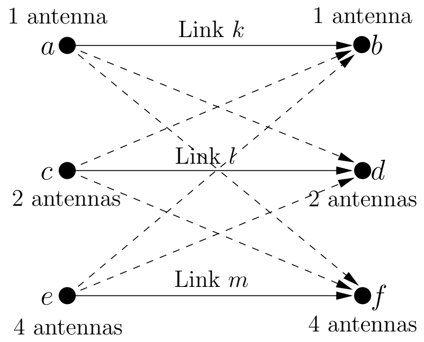
$$z_l + \sum_{k \in \mathcal{I}_l}^{\Pi(\text{Tx}(l)) > \Pi(\text{Rx}(k))} z_k \le \alpha . \tag{6}$$

2) DoF constraint at Rx(l): The number of DoFs that receiver Rx(l) can use for SM (for reception) and IC cannot exceed the total number of DoFs at node Rx(l), i.e.,

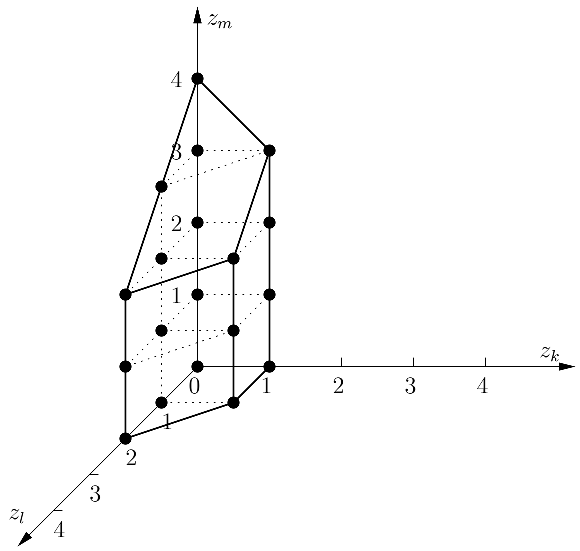
$$z_l + \sum_{k \in \mathcal{Q}_l}^{\Pi(\text{Rx}(l)) > \Pi(\text{Tx}(k))} z_k \le \alpha . \tag{7}$$

We use the following simple example to illustrate DoF allocation in a MIMO network.

Example 2. Consider the three-link (k, l, and m) example in Fig. 6(a). The number of antennas at each node is also shown in the figure. Under the above MIMO model, we need an order to determine the DoF resource usage at each node. Suppose we are following an order list, say $a \to d \to b \to c \to e \to f$



(a) Inter-nodal interference relationship for three links.



(b) Achievable DoF region of the three MIMO links.

Fig. 6. A three-link MIMO network example.

among the nodes. Then, the DoF allocation in this MIMO network works as follows.

We start with node a, which is the first node in the list. Given it is the first in the list, node a does not have any interference with which it needs to be concerned. Since node a has only 1 antenna, it can transmit at most 1 data stream to its intended receiver b. The second node on the ordered list is node d. Since it appears in the order list after node a, node d needs to suppress the interference from a. This implies that node d needs to expend 1 DoF to cancel the interference from a. Since d has 2 antennas, we have that d can receive at most $2 - 1 = 1$ stream, i.e., $z_l \leq 1$. The DoF consumption on nodes b and c follows exactly the same token, and it can be verified that b and c can each receive and transmit 1 stream, respectively. Since node e's transmission should not interfere with the reception at b and d that had appeared in the order list earlier, e needs to expend 2 DoFs for this purpose. At this point, e can transmit up to $4 - 1 - 1 = 2$ streams, i.e., $z_m \leq 2$. Finally, along the same line, node f can receive at most $4 - 1 - 1 = 2$ streams, i.e., $z_m \leq 2$. Therefore, after the above steps, we can see that the stream combination $(z_k = 1, z_l = 1, z_m = 2)$ can be scheduled feasibly on links k, l, and m. It can be shown that the entire DoF region (the set of all feasible stream combinations) for the three-link example in Fig. 6(a) can be found by enumerating all possible choices of the node order list. Each stream combination offers a feasible point (e.g., $(1, 1, 2)$), the union of which constitutes the DoF region, which we plot in Fig. 6(b).

6.2 Calculating $f_{\text{RX}}(n)$

Based on the MIMO network model, we now calculate $f_{\text{RX}}(n)$.[5] Recall that $f_{\text{RX}}(n)$ is an upper bound of the maximum number of successful transmissions whose receivers are in the same interference square. In the case of MIMO, this corresponds to the maximum number of successful data streams on all active links whose receivers are in the same interference square.

Lemma 5. For a random MIMO ad hoc network, we have $f_{\text{RX}}(n) = \alpha$.

Proof: Denote $\mathcal{L}$ the set of active links with their receivers being in the same interference square. Denote $|\mathcal{L}|$ the number of links in $\mathcal{L}$, and let $\mathcal{L} = \{1, \ldots, |\mathcal{L}|\}$. Our goal is to find an upper bound for the sum of data streams on these links, i.e., $\sum_{k \in \mathcal{L}} z_k$.

If $|\mathcal{L}| = 1$, i.e., only one active link with its receiver in the interference square, then $z_1 \leq \alpha$ (since the number of data streams on this link cannot exceed the number DoFs of a node). We can set $f_{\text{RX}}(n) = \alpha$ and the lemma holds trivially.

For the general case of $|\mathcal{L}| \geq 2$, Property 1 says that these $|\mathcal{L}|$ links interfere with each other and IC is necessary. Based on the MIMO model we discussed earlier, we need to follow an ordered list for the nodes (both transmitters and receivers) on these $|\mathcal{L}|$ links for DoF allocation at each node. We have two cases, depending on whether the last node in the list is a transmitter or a receiver.

Case (i). The last node in the ordered list is a receiver. Without loss of generality, denote m as the link of which this node is the receiver. To have z_m data streams on link m, based on (7), we have the following constraint on receiver $\text{Rx}(m)$.

$$z_m + \sum_{k \in \mathcal{Q}_m}^{\Pi(\text{Rx}(m)) > \Pi(\text{Tx}(k))} z_k \leq \alpha \ , \tag{8}$$

where the sum for z_k is taken over all interfering links whose transmitters are before receiver $\text{Rx}(m)$ in the node list. Since link m is being interfered by all other links in $\mathcal{L}$ in the same interference square, we have $\mathcal{Q}_m = \mathcal{L} \backslash \{m\}$. Further, since $\text{Rx}(m)$ is the last node in this list, we have $\Pi(\text{Rx}(m)) > \Pi(\text{Tx}(k))$, for all $k \in \mathcal{L} \backslash \{m\}$. Therefore, (8) can be re-written as

$$z_m + \sum_{k \in \mathcal{L} \backslash \{m\}} z_k \leq \alpha \ ,$$

which is

$$\sum_{k \in \mathcal{L}} z_k \leq \alpha \ .$$

Thus, we have shown that the sum of data streams that can be received by nodes in the interference square over all links is upper bounded by α, i.e., $f_{\text{RX}}(n) = \alpha$.

Case (ii). The last node in the ordered list is a transmitter. In this case, we employ (6) and follow the same token as the above discussion. We again have $f_{\text{RX}}(n) = \alpha$.

Combining the two cases, we have $f_{\text{RX}}(n) = \alpha$. □

5. For MIMO, the level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as $f_{\text{TX}}(n)$ and either approach will yield the same result.

6.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = \alpha$, we can now apply Theorem 1 and obtain asymptotic capacity upper bound of a random MIMO ad hoc network as follows.

Proposition 3. For a random MIMO ad hoc network, we have $\lambda(n) = O(1/\sqrt{n \ln n})$ almost surely when $n \to \infty$.

This result is the same as that in [10]. This upper bound is tight since it has the same asymptotic order as the lower bound shown in [10]. It is also interesting to see that, despite MIMO's ability to increase capacity in a network with finite number of nodes, the scaling order for its asymptotic capacity remains the same as that for a single omnidirectional antenna network as in [7]. Finally, the advantage of our approach is that its analysis is much simpler than that in [10]. Such advantage also holds in the following sections for other advanced physical layer techniques.

7 CASE STUDY III: MULTI-CHANNEL AND MULTI-RADIO

7.1 Multi-Channel Multi-Radio Model

Multi-channel multi-radio (MC-MR) refers that there are multiple channels in the network and there are multiple radio interfaces at each node in the network [12], [13]. By equipping each node with multiple radio interfaces, each node has more flexibility in channel access in the network. The protocol model in MC-MR is defined as follows. Following [12], we assume that there are c channels in the network and each node in the network is equipped with m radio interfaces, where c and m are constants, and $1 \le m \le c$. A radio interface is capable of transmitting or receiving data on only one channel at any given time, i.e., half-duplex.

- On a specific channel, a transmitting radio can send data only to a receiving radio within its transmission range.
- Other transmitting radios must be out of the interference range of this receiving radio.

7.2 Calculating $f_{\text{RX}}(n)$

Based on the MC-MR model, we now calculate $f_{\text{RX}}(n)$.[6] Assuming each band has the same bandwidth in the MC-MR network, then $f_{\text{RX}}(n)$ corresponds to the maximum number of successful transmissions over all available channels on all radio interfaces whose receivers are in the same interference square. We have the following lemma.

Lemma 6. For a random MC-MR network, we have $f_{\text{RX}}(n) = c$.

Proof: Let's focus on one channel at a time. Since the links with receivers in the interference square interfere with each other (Property 1), there can be at most one radio at a node receiving on this channel. Summing up all such radios (or successful transmissions) over c channels, we have $f_{\text{RX}}(n) = c$. □

6. For an MC-MR network, the level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as $f_{\text{TX}}(n)$ and either approach will yield the same result.

7.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = c$, we can now apply Theorem 1 and obtain asymptotic capacity upper bound of an MC-MR ad hoc network as follows.

Proposition 4. For a random MC-MR ad hoc network, we have $\lambda(n) = O\left(1/\sqrt{n \ln n}\right)$ almost surely when $n \to \infty$.

Note that this result is the same as the result in [12] for the case when $\frac{c}{m} = O(\ln n)$. This upper bound is tight since it has the same asymptotic order as the lower bound shown in [12].

8 CASE STUDY IV: COGNITIVE RADIO AD HOC NETWORKS

8.1 Cognitive Radio Network Model

Cognitive radio (CR) is another new physical layer technology that enables more efficient utilization of radio spectrum [23]. A CR is able to constantly sense the radio spectrum and explore any available spectrum bands for data communication. Consider a random ad hoc network where each node is equipped with a CR. Consider a specific time instance where each node i senses a set of available frequency bands $\mathcal{B}_i$ that it can use.[7] Note that due to differences in locations, the set of available frequency bands $\mathcal{B}_i$ at a node i may be different from that at another node in the network. Denote $\mathcal{B}_{ij} = \mathcal{B}_i \bigcap \mathcal{B}_j$ the set of common bands that are available at both nodes i and j. Then node i can communicate to node j on band m only if $m \in \mathcal{B}_{ij}$. The protocol model for CR is defined as follows. Node i can successfully communicate to node j on band m if and only if

- band m is the common band of both node i and node j;
- node j is within the transmission range of node i;
- node j is outside the interference range of other non-intended transmitters.

8.2 Calculating $f_{\text{RX}}(n)$

Based on the CR network model, we now calculate $f_{\text{RX}}(n)$.[8] Assuming that each band has the same bandwidth in the CR network, then $f_{\text{RX}}(n)$ corresponds to the maximum number of successful transmissions over all available bands whose receivers are in the same interference square. Denote $M = |\bigcup_{i=1}^{n} \mathcal{B}_i|$, i.e., M is the number of distinct frequency bands in the network. Then we have the following lemma.

Lemma 7. For a random CR ad hoc network, we have $f_{\text{RX}}(n) = M$.

Proof: Consider one band at a time. Within each band, by Property 1, the links with receivers in the interference square interfere with each other. So the maximum number of active links (or successful transmissions) is at most one. Summing up all active links (or successful transmissions) over M bands, we have $f_{\text{RX}}(n) = M$. □

7. These bands may be those that are currently unused by the primary users.
8. For a CR network, the level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as $f_{\text{TX}}(n)$ and either approach will yield the same result.

8.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = M$, we can now apply Theorem 1 and obtain asymptotic capacity upper bound for a random CR ad hoc network as follows.

Proposition 5. For a random CR ad hoc network, we have $\lambda(n) = O\left(1/\sqrt{n \ln n}\right)$ almost surely when $n \to \infty$.

This result is consistent to those found in [8], [18]. This upper bound is tight since it has the same asymptotic order as the lower bound shown in [8], [18].

9 Case Study V: Ad Hoc Networks with Multi-Packet Reception

Multi-packet reception (MPR) is a conceptual abstraction of a physical layer capability that a receiver can correctly decode multiple packets from different transmitters simultaneously [21]. As described in [16], such capability may be implemented by a variety of advanced physical layer technologies, such as multiuser detection [22], directional antenna [15], [25], and MIMO. In other words, MPR refers to a reception capability of a node at the physical layer, rather than referring to a specific physical layer technology. In this section, we employ our criteria in Section 4 to explore capacity scaling law of MPR-based ad hoc networks.

9.1 A General MPR Model

Under MPR, a transmitter can transmit packet to only one receiver at a time, but a receiver is capable of receiving multiple packets simultaneously from multiple transmitters within its transmission range. For unintended transmissions whose interference range covers a receiver, the receiver will consider them as interference. Such interference may be cancelled by the receiver. Specifically, in the MPR model, we assume a receiver has finite resource available for MPR and interference cancellation. Denote β_1 the number of simultaneous packets from intended transmitters whose transmission range covers the receiver and β_2 the number of unintended transmitters that produce interference on the same receiver. We have

$$\beta_1 + \beta_2 \le \beta ,$$

where β is a constant and represents the total available resource at a receiver. For example, if MIMO is employed to implement MPR, then the number of DoFs at a MIMO node may correspond to β.

Note that this MPR model is a generalization of the idealized MPR model in [16] which assumes $\beta_1 \le \beta = \infty$ and $\beta_2 = 0$, i.e., a receiver can successfully decode arbitrary number of simultaneous packet transmissions and no interference is allowed on the receiver.

9.2 Calculating $f_{\text{RX}}(n)$

We choose to calculate $f_{\text{RX}}(n)$, which is more convenient than calculating $f_{\text{TX}}(n)$. In the case of MPR ad hoc networks, $f_{\text{RX}}(n)$ corresponds to an upper bound of the maximum number of packets that are successfully received simultaneously by all the receivers in the same interference square. We have the following lemma for $f_{\text{RX}}(n)$.

Lemma 8. For a random MPR ad hoc network, we have $f_{\text{RX}}(n) = \beta$.

Proof: Denote $\mathcal{L}$ the set of successful links with their receivers residing in the same interference square. By a “successful” link, we mean the receiver of this link can successfully decode the packet on this link. Denote $|\mathcal{L}|$ the number of links in $\mathcal{L}$, and let $\mathcal{L} = \{1, \ldots, |\mathcal{L}|\}$. Then $f_{\text{RX}}(n)$ is an upper bound of $|\mathcal{L}|$.

Note that for two successful links, their transmitters are different but their receivers may be the same. Consider one receiver j in the interference square. From receiver j's perspective, we divide $\mathcal{L}$ into two subsets: $\mathcal{L}_1$ — the set of links whose receivers are j, and $\mathcal{L}_2$ — the set of links whose receivers are not j. Based on Property 1, we know that the transmitters of the links in subset $\mathcal{L}_2$ are all in the interference range of receiver j. Since packets on $\mathcal{L}_1$ are successfully received by j, then based on the MPR model, we have

$$|\mathcal{L}| = |\mathcal{L}_1| + |\mathcal{L}_2| = \beta_1 + \beta_2 \le \beta .$$

Therefore, we have $f_{\text{RX}}(n) = \beta$. □

9.3 Scaling Law

Following Fig. 4, with $f_{\text{RX}}(n) = \beta$, we can now apply Theorem 1 and directly obtain the following asymptotic capacity upper bound for an MPR-based ad hoc network.

Proposition 6. For a random MPR ad hoc network, we have $\lambda(n) = O(1/\sqrt{n \ln n})$ almost surely when $n \to \infty$.

The above upper bound for MPR is a new result obtained via our unified approach.

9.4 An Idealized MPR Model

For the idealized MPR model described in [16], where $\beta_1 \le \beta = \infty$ and $\beta_2 = 0$, one can still apply our simple scaling order criteria. In particular, it can be shown that for this idealized MPR model, we have $f_{\text{RX}}(n) = \Theta\left(nr^2(n)\right)$ in Lemma 9.

Lemma 9. For a random ad hoc network under the idealized MPR model, we have $f_{\text{RX}}(n) = \Theta(nr^2(n))$.

Proof: First, we show that there can be only one receiver (say j) in the interference square receiving packets. This can be shown by contradiction. Suppose there is another receiver i, $i \neq j$, that receives packets in the same interference square. Then, based on Property 1, one of receiver i's transmitters must be within the interference range of node j. This transmitter of receiver i will interfere node j, which contradicts with $\beta_2 = 0$ under the idealized MPR model.

Based on Lemma 3, we know that the number of all nodes inside the larger square is $\Theta(nr^2(n))$. Since each transmitter transmits one packet to receiver j at a time, the number of simultaneous packets received by receiver j cannot exceed the number of nodes in the larger square, i.e., $\Theta(nr^2(n))$. Therefore, we have $f_{\text{RX}}(n) = \Theta(nr^2(n))$. □

Combining Lemma 9 and Theorem 1, we have

$$\lambda(n) = O\left(\frac{f_{\text{RX}}(n)}{nr(n)}\right) = O\left(nr^2(n) \cdot \frac{1}{nr(n)}\right) = O\left(r(n)\right) .$$

This is exactly the result developed in [16]. This upper bound is tight since it has the same asymptotic order as the lower bound shown in [16].

TABLE 2
A summary of asymptotic capacity upper bounds obtained via our simple criteria. "—" sign indicates new result not available in literature.

Physical layer technology		$f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$	Upper bound	Reference
Directional antenna	Single beam	$f_{\text{TX}}(n) = \Theta\left(nr^2(n)\right)$	$O\left(r(n)\right)$	[15]
	Multi-beam	$f_{\text{RX}}(n) = O\left(n^2r^4(n)\right)$	$O\left(nr^3(n)\right)$	[15]
MIMO		$f_{\text{RX}}(n) = \alpha$	$O\left(\frac{1}{\sqrt{n\ln n}}\right)$	[10]
MC-MR		$f_{\text{RX}}(n) = c$	$O\left(\frac{1}{\sqrt{n\ln n}}\right)$	[12]
CR		$f_{\text{RX}}(n) = M$	$O\left(\frac{1}{\sqrt{n\ln n}}\right)$	[8], [18]
MPR	General	$f_{\text{RX}}(n) = \beta$	$O\left(\frac{1}{\sqrt{n\ln n}}\right)$	—
	Idealized	$f_{\text{RX}}(n) = \Theta\left(nr^2(n)\right)$	$O\left(r(n)\right)$	[16]
Full-duplex		$f_{\text{RX}}(n) = 2$	$O\left(\frac{1}{\sqrt{n\ln n}}\right)$	[24]

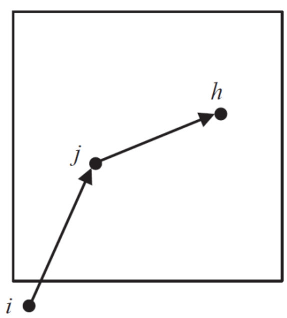


Fig. 7. Two transmissions under full-duplex whose receivers are in the same interference square.

10 Case Study VI: Full-Duplex Radio

10.1 Full-Duplex Model

Full-duplex refers that a radio can transmit and receive different packets at the same time on the same channel [24]. When node i transmits to node j, the necessary and sufficient conditions for a successful transmission (allowing full-duplex) are:

- node j is within the transmission range of node i, i.e., $d_{ij} \le r(n)$, where d_{ij} is the distance between nodes i and j, and
- for any transmitting node k other than nodes i and j, node j is outside the interference range of node k, i.e., $d_{kj} > (1+\Delta)r(n)$, $k \ne i, j$.

This protocol model for full-duplex is similar to the protocol model in half-duplex, except that we have $k \ne j$ when we list constraints in the second condition.

A full-duplex example is shown in Fig. 7, where there are two transmissions $i \to j$ and $j \to h$ and we have $d_{ij} < r(n), d_{jh} < r(n), d_{ih} > (1+\Delta)r(n)$. We now show that all full-duplex constraints are satisfied for these two transmissions. For transmission $i \to j$, the first condition requires $d_{ij} \le r(n)$, which is satisfied. The second condition requires that we consider any transmitting node k other than nodes i and j, which is an empty set, i.e., there is no constraint posed by the second condition. For transmission $j \to h$, the first condition requires $d_{jh} \le r(n)$, which is satisfied. The second condition requires that we consider any transmitting node k other than nodes j and h. Since node i is the only such transmitting node, i.e., the second condition requires that $d_{ih} > (1+\Delta)r(n)$, which is satisfied. Therefore, all full-duplex constraints are satisfied for this example and we have full-duplex at node j.

10.2 Calculating $f_{\text{RX}}(n)$

Based on the full-duplex model, we now calculate $f_{\text{RX}}(n)$.[9] Suppose that in an interference square, there is a successful transmission from node j to node h with both nodes in this interference square. With full-duplex at node j, we can have at most another successful transmission from node i to node j (see Fig. 7). Thus, we have the following lemma.

Lemma 10. For a random full-duplex network, we have $f_{\text{RX}}(n) = 2$.

10.3 Scaling Law

Following Lemma 10, with $f_{\text{RX}}(n) = 2$, we can now apply Theorem 1 and obtain asymptotic capacity upper bound of a full-diplex ad hoc network as follows.

Proposition 7. For a random full-duplex ad hoc network, we have $\lambda(n) = O\left(1/\sqrt{n\ln n}\right)$ almost surely when $n \to \infty$.

Note that this result is the same as the result in [24]. Since a half-duplex feasible solution is also a feasible solution for a full-duplex network, we can use the lower bound $\Omega\left(1/\sqrt{n\ln n}\right)$ developed in [7] as a lower bound for a full-duplex network. Then the above upper bound is tight since it has the same asymptotic order as the lower bound.

11 Discussions

11.1 Summary of Results

Table 2 summarizes asymptotic capacity upper bounds that we derived in the last six sections by applying our proposed new method. For the MPR general model, the result that we developed in this paper is new and not available in the literature. Note that our results are consistent to those reported in the literature (last column of Table 2), each of which was found via custom-designed and complex mathematical analysis. In contrast, the method we used to develop these bounds is simple and general. It serves not only as a simple tool to validate the capacity bound under those PHY technologies in [8], [10], [12], [15], [16], [18], [24], but also offer a powerful tool to determine capacity bounds under other PHY technologies in the future.

We caution that the success of our simple method hinges upon the calculation of $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$. One should calculate $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$ as tight as possible since loose $f_{\text{RX}}(n)$ or $f_{\text{TX}}(n)$ (e.g., infinity) will yield trivial upper bounds.

9. For a full-duplex network, the level of difficulty in calculating $f_{\text{RX}}(n)$ is the same as $f_{\text{TX}}(n)$ and either approach will yield the same result.

11.2 Asymptotic Order Change

We observe that for advanced PHY technologies such as MIMO, MC-MR, cognitive radio, general MPR, and full-duplex, the asymptotic capacity upper bounds are $O\left(1/\sqrt{n\ln n}\right)$, which is the same as that under single omnidirectional antenna [7]. Given that $O\left(1/\sqrt{n\ln n}\right)$ is a tight upper bound, we conclude MIMO, MC-MR, cognitive radio, general MPR, and full-duplex cannot make fundamental change in asymptotic order.[10] This is an interesting result. On the other hand, under directional antenna and idealized MPR, the asymptotic capacity upper bounds are on a higher order than $O\left(1/\sqrt{n\ln n}\right)$. This indicates that the latter PHY technologies have potential to improve network capacity in the asymptotic sense.

12 Conclusions

In this paper, we presented a simple yet powerful method that one can apply to quickly determine the asymptotic capacity bounds under the protocol model for various PHY layer technologies. This new method offers a general tool to determine capacity scaling law, which is in contrast to existing approaches, which were based on complex mathematical analysis that was custom-designed for each PHY technology. We proved the correctness of our proposed method and demonstrated its applications through a number of case studies, such as wireless networks with directional antenna, MIMO, MC-MR, cognitive radio, MPR, and full-duplex radio. The new method in this paper offers a simple tool to wireless networking researchers to quickly understand asymptotic capacity of wireless networks under a particular PHY layer technology.

An open problem is whether a simple method like ours also exists for SINR-based (physical) interference models, in addition to the protocol model. After a number of attempts, we conjecture that this is not possible. This is because, a successful transmission under the SINR-based model requires complex calculation of SINR at a receiver, which cannot be handled by distance-based accounting of interfering nodes. Even worse, there does not even appear to exist a general SINR-like physical model that can accommodate different PHY layer technologies (e.g., MIMO, directional antenna, MPR), which is necessary to develop a general method to analyze capacity bounds. Due to these fundamental difficulties and after our rather thorough investigation through different avenues, we believe that a simple method like ours is unlikely to exist in the world of SINR-based interference models. We leave it as a conjecture for future research.

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10. It is important to realize that capacity scaling law only shows a general trend on how capacity changes when $n \to \infty$. Therefore, no improvement in asymptotic capacity does not mean there is no improvement in capacity when network size is finite. It is well known that most of these advanced physical layer technologies can significantly improve network capacity in finite-sized networks.

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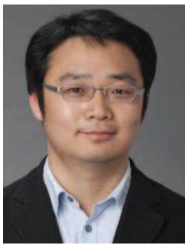

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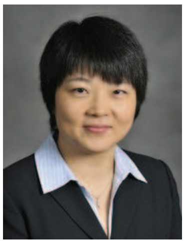

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