Weighted Proportional Fairness Capacity of Gaussian MIMO Broadcast Channels

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Abstract—**Recently, there has been tremendous interest in exploring the capacity region of multiple-input multiple-output broadcast channels (MIMO-BC). However, fairness, a very important performance measure of multi-user communications systems and networks, has not been addressed for MIMO-BC in the literature. In this paper, we study how to determine the weighted proportional fairness (WPF) capacity of MIMO-BC. The difficulty of finding the WPF capacity of MIMO-BC lies in that it contains two difficult subproblems: 1) a complex combinatorial optimization problem to determine the optimal decoding order in the dual MIMO multiple access channel (MIMO-MAC) and 2) a nonconvex optimization problem in computing the optimal input covariance matrices to achieve WPF capacity. To circumvent the difficulty in the first subproblem, we derive a set of optimality conditions that the optimal decoding order must satisfy. Based on these optimality conditions, we design an efficient algorithm called iterative gradient sorting (IGS) to determine the optimal decoding order by iteratively sorting the gradient entries and moving across corner points. We also show that this method can be geometrically interpreted as sequential gradient projections. For the second subproblem, we propose an efficient algorithm based on conjugate gradient projection (CGP) technique, which employs the concept of Hessian conjugate. We also develop a polynomial time algorithm to solve the projection subproblem.**

I. Introduction

In network information theory, a MIMO broadcast channel (MIMO-BC) refers to a communication system where a single multi-antenna transmitter sends independent information to multiple *uncoordinated* multi-antenna receivers. MIMO-BC belongs to the class of nondegraded broadcast channels, for which the capacity region is notoriously difficult to analyze [1]. Over the years, characterizing the capacity region of MIMO-BC has been one of the most fundamental problems in network information theory.

Recently, Weigarten *et al.* [2] proved the long-open conjecture that "dirty paper coding" (DPC) achieves the entire capacity region of MIMO-BC. Moreover, by the *channel duality* between MIMO-BC and MIMO multiple access channel (MIMO-MAC) [3]–[5], the complex DPC rate region of MIMO-BC can be analyzed using its dual MIMO-MAC under a sum power constraint. Although DPC is known to be the optimal transmission strategy for MIMO-BC, many optimization problems of MIMO-BC over DPC rate region remain unsolved even with the channel duality transformation. So far, only the maximum weighted sum rate (MWSR) problem is solved in the literature. This is because determining the optimal decoding order in the dual MIMO-MAC is relatively simple due to the linearity of the MWSR's objective function. As a result, MWSR can be transformed into a convex optimization problem, which can be solved efficiently [6]–[9].

However, the maximum weighted sum rate-based objective function may entail fairness issues among users. In practice, fairness is a key performance measure in a multi-user communication system. A widely accepted objective is the so-called *weighted proportional fairness* (WPF), which is introduced by Kelly *et al.* [10]. The problem of MIMO-BC WPF capacity arises naturally from designing a proportional-fair scheduler for the downlinks of cellular systems or performing cross-layer optimization for MIMO-BC based mesh networks with random access (see [11] for more details).

For a K-user MIMO-BC, the WPF objective function can be written as $\sum_{i=1}^{K} w_i \log R_i$, where $w_i > 0$ and $R_i > 0$ are the weight and the rate of user i, respectively. Despite the simplicity of the WPF objective function, determining the WPF capacity for MIMO-BC turns out to be a surprisingly hard problem. First, the nonlinearity of the WPF objective function makes it very difficult to determine the optimal decoding order in its dual MIMO-MAC. Without any smart algorithm, we may have to enumerate and compare all $K!$ corner points of the capacity region of the dual MIMO-MAC, each of which corresponds to one particular decoding order. As a result, one can be trapped in an intractable combinatorial optimization problem. Even worse, in some cases, an optimal decoding order may not even exist and as a result, it is impossible to detect a nonexistent solution by just enumerating and comparing the corner points. Second, even if we have the knowledge of the optimal decoding order, the WPF problem still cannot be simplified into a convex optimization problem. This is because the interference terms in the rate expressions of the corner points in the dual MIMO-MAC capacity region cannot be canceled out in the WPF objective function, which is in the form of sum of weighted logs.

In this paper, we aim to tackle this difficult problem of determining the WPF capacity for MIMO-BC. Our approach consists of the following two steps: 1) determine the optimal decoding order in the dual MIMO-MAC or show its nonexistence; and 2) compute the optimal input covariance matrices to achieve the WPF capacity under the optimal decoding order. The main contributions of this paper are three-fold:

1) Based on the special structure of the corner points in the dual MIMO-MAC capacity region, we develop a set of necessary and sufficient *optimality conditions* for the

optimal decoding order in the dual MIMO-MAC. For the case where the optimal decoding order does not exist, we derive a set of closed-form expressions to quickly compute the optimal WPF rates of MIMO-BC.
2) Based on the optimality conditions we derive, we design an efficient algorithm called iterative gradient sorting (IGS) to determine the optimal decoding order by iteratively sorting the gradient entries and moving across corner points. We show that this method can be geometrically interpreted as sequential gradient projections.
3) We design an efficient algorithm called *conjugate gradient projection* (CGP) to compute the optimal input covariance matrices so as to achieve the WPF capacity. In CGP, we use *conjugate* gradient directions to eliminate the "zigzagging" phenomenon so that CGP can achieve a superlinear convergence rate.

The remainder of this paper is organized as follows. In Section II, we describe the network model and formulate our problem. Section III introduces some important concepts of the dual MIMO-MAC capacity region. In Section IV, we derive the optimality conditions for the optimal decoding order and discuss its geometrical insights. Section V introduces how to determine the optimal decoding order based on the optimality conditions and its geometrical interpretations. In Section VI, we investigate how to compute optimal input covariance matrices under the optimal decoding order. Section VII concludes this paper.

II. System Model and Problem Formulation

We first introduce notation for matrices, vectors, and complex scalars in this paper. We use boldface to denote matrices and vectors. For a matrix $\mathbf{A}$, $\mathbf{A}^\dagger$ denotes the conjugate transpose, $\mathrm{Tr}\{\mathbf{A}\}$ denotes the trace of $\mathbf{A}$, and $|\mathbf{A}|$ denotes the determinant of $\mathbf{A}$. $\mathrm{Diag}\{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ represents the block diagonal matrix with matrices $\mathbf{A}_1, \ldots, \mathbf{A}_n$ on its main diagonal. We let $\mathbf{I}$ denote the identity matrix with its dimension determined from the context. $\mathbf{A} \succeq 0$ represents that $\mathbf{A}$ is Hermitian and positive semidefinite (PSD). $(\mathbf{v})_m$ represents the m^{th} entry of vector $\mathbf{v}$. We let $\mathbf{e}_i$ be the unit column vector where the i^{th} entry is 1 and all other entries are 0. The dimension of $\mathbf{e}_i$ is determined from the context. We let $\mathbf{1}$ and $\mathbf{0}$ be the column vector where all entries are equal to 1 and 0, respectively. The dimensions of $\mathbf{1}$ and $\mathbf{0}$ are determined from the context. The operator "$\langle \cdot, \cdot \rangle$" represents the inner product operation for vectors or matrices.

Suppose that a MIMO-BC has K users, each of which is equipped with n_r antennas, and the transmitter has n_t antennas. The channel gain matrix for user i is denoted as $\mathbf{H}_i \in \mathbb{C}^{n_r \times n_t}$. In [2], it has been shown that the capacity region of MIMO-BC is equal to the DPC rate region. In the DPC rate region, suppose that users $1, \ldots, K$ are encoded sequentially. Then the rate of user i can be computed as: [3]

$$R_i^{\mathrm{DPC}}(\mathbf{\Gamma}) = \log \frac{\left|\mathbf{I} + \mathbf{H}_i \left(\sum_{j=i}^{K} \mathbf{\Gamma}_j\right) \mathbf{H}_i^\dagger\right|}{\left|\mathbf{I} + \mathbf{H}_i \left(\sum_{j=i+1}^{K} \mathbf{\Gamma}_j\right) \mathbf{H}_i^\dagger\right|}, \qquad (1)$$

where $\mathbf{\Gamma}_i \in \mathbb{C}^{n_t \times n_t}$, $i = 1, \ldots, K$, are the *downlink* input covariance matrices, and where $\mathbf{\Gamma} \triangleq \{\mathbf{\Gamma}_1, \ldots \mathbf{\Gamma}_K\}$ denotes the collection of all downlink covariance matrices. As a result, the WPF problem can be written as follows:

$$\begin{array}{ll} \text{Maximize} & \sum_{i=1}^{K} w_i \log R_i^{\mathrm{DPC}}(\mathbf{\Gamma}) \\ \text{subject to} & \mathbf{\Gamma}_i \succeq 0, \quad i = 1, \ldots, K \\ & \sum_{i=1}^{K} \mathrm{Tr}(\mathbf{\Gamma}_i) \leq P, \end{array} \qquad (2)$$

where w_i is the weight of user i, P represents the maximum transmit power at the transmitter. Furthermore, due to the duality between MIMO-BC and MIMO-MAC [3], the rates achievable in a MIMO-BC are also achievable in its dual MIMO-MAC. That is, given a feasible $\mathbf{\Gamma}$, there exists a set of feasible *uplink* input covariance matrices for the dual MIMO-MAC, denoted by $\mathbf{Q}$, such that $R_i^{\mathrm{MAC}}(\mathbf{Q}) = R_i^{\mathrm{DPC}}(\mathbf{\Gamma})$. As a result, the complex MIMO-BC capacity region can be transformed into its dual MIMO-MAC, for which the capacity region is much easier to analyze [1]. Thus, (2) is equivalent to the following WPF problem of the dual MIMO-MAC:

$$\begin{array}{ll} \text{Maximize} & \sum_{i=1}^{K} w_i \log R_i^{\mathrm{MAC}}(\mathbf{Q}) \\ \text{subject to} & \mathbf{R}^{\mathrm{MAC}}(\mathbf{Q}) \in \mathcal{C}_{\mathrm{MAC}}(P, \mathbf{H}^\dagger), \ i = 1, \ldots, K, \end{array} \qquad (3)$$

where $\mathbf{R}^{\mathrm{MAC}}(\mathbf{Q}) \triangleq \{R_i^{\mathrm{MAC}}(\mathbf{Q}) : i = 1, \ldots, K\}$ is a collection of all users' data rates in the dual MIMO-MAC, and $\mathcal{C}_{\mathrm{MAC}}(P, \mathbf{H}^\dagger)$ represents the capacity region of the dual MIMO-MAC. For simplicity, we drop the superscript "MAC" and simply refer to user i's rate in the dual MIMO-MAC as R_i. From [1, Theorem 14.3.5], it can be shown that $\mathcal{C}_{\mathrm{MAC}}(P, \mathbf{H}^\dagger)$ is determined by

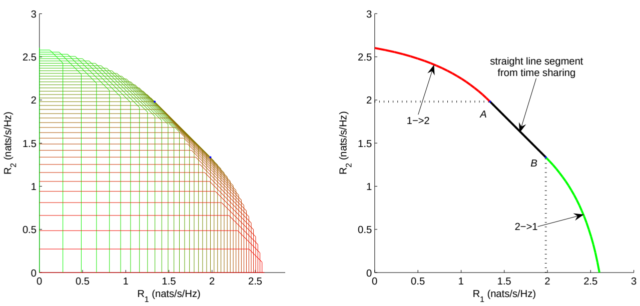
$$\mathcal{C}_{\mathrm{MAC}}(P, \mathbf{H}^\dagger) = Conv \left\{ (R_1, \ldots, R_K) \;\middle|\; \begin{array}{l} \sum_{i \in \mathcal{S}} R_i(\mathbf{Q}) \leq \\ \log \left| \mathbf{I} + \sum_{i \in \mathcal{S}} \mathbf{H}_i^\dagger \mathbf{Q}_i \mathbf{H}_i \right|, \\ \forall \mathcal{S} \subseteq \{1, \ldots, K\}, \\ \sum_{i=1}^{K} \mathrm{Tr}(\mathbf{Q}_i) \leq P, \\ \mathbf{Q}_i \succeq 0, \ \forall i. \end{array} \right\}, \qquad (4)$$

where $Conv(\cdot)$ represents the convex hull operation, $\mathbf{Q}_i \in \mathbb{C}^{n_r \times n_r}$, $i = 1, \ldots, K$, are the uplink input covariance matrices. If the dual MIMO-MAC is Gaussian, the convex hull operation can be dropped [1].

III. Capacity Region of the dual MIMO-MAC

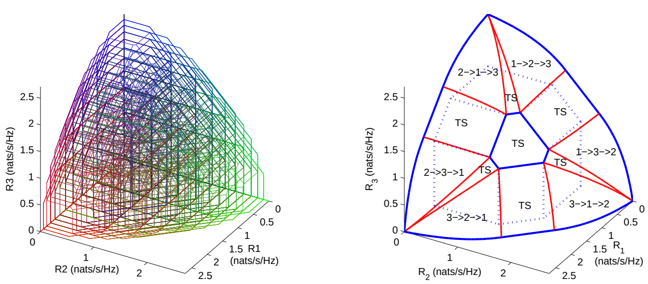
Since we will study the WPF capacity for MIMO-BC based on its dual MIMO-MAC, it is beneficial to characterize the capacity region of the dual MIMO-MAC channel first. It can be seen from (4) that the capacity region of a K-user dual MIMO-MAC has a polymatriod structure with $2^K - 1$ sum rate constraints in total. In general, the region defined by (4) for a fixed set of transmit covariance matrices $\mathbf{Q}$ is a beveled box with $2^K - 1$ faces. The capacity region of a dual MIMO-MAC is the union of all beveled boxes for all feasible input covariance matrices. For example, Fig. 1(a) and Fig. 2(a) show the capacity region of a two-user and a three-user dual MIMO-MAC, respectively. For easier visualization, we plot the boundaries of the respective capacity regions in

Fig. 1(b) and Fig. 2(b). As shown in Fig. 1(b), the boundary of the two-user dual MIMO-MAC capacity region contains two curves that correspond to decoding order $1 \to 2$ and $2 \to 1$, respectively, and a straight line segment represents time sharing between corner points A and B, which achieve the maximum sum rate with user 1 and user 2 being decoded first, respectively. Similarly, as shown in Fig. 2(b), there are six subregions on the boundary corresponding to $3! = 6$ decoding orders. Also, there exists a hexagon defined by six corner points and six other subregions of surfaces on the boundary that are achieved by time sharing (labeled as “TS” in Fig. 2(b)).



(a) Union of 2D beveled boxes. (b) The boundary.

Fig. 1. An example of the capacity region of a two-user dual MIMO-MAC: $\mathbf{H}_1 = [1\ 0.5]$, $\mathbf{H}_2 = [0.5\ 1]$, and $P = 10$.



(a) Union of 3D beveled boxes. (b) The boundary.

Fig. 2. An example of the capacity region of a three-user dual MIMO-MAC: $\mathbf{H}_1 = [1\ 0.4\ 0.5]$, $\mathbf{H}_2 = [0.5\ 1\ 0.4]$, $\mathbf{H}_3 = [0.4\ 0.5\ 1]$, and $P = 10$.

In general, we have the following lemma to count the number of time-sharing subregions on the boundary.

Lemma 1. *For a K-user dual MIMO-MAC, the total number of time-sharing subregions on the boundary is given by $\sum_{i=2}^{K} \binom{K}{i}(K-i+1)!$.*

Proof. For a time-sharing subregion involving i users in a decoding order, there are $\binom{K}{i}$ possible combinations in total. Treating those i users as a single element in an ordering, we have $(K-i+1)!$ permutations. Thus, the total number of time-sharing subregions is $\sum_{i=2}^{K} \binom{K}{i}(K-i+1)!$. $\square$

It can be seen that the decoding orders have significant impact on the objective value of (3). Some of them can achieve better objective values than the others. So, we introduce the following definition.

Definition 1 (Optimal Decoding Order and Optimal Corner Point)**.** *A decoding order $\pi(\cdot)$ is an optimal decoding order of the dual MIMO-MAC if an optimal solution of (3) is achieved at a point on the subregion of the boundary that corresponds to $\pi(\cdot)$. The corresponding corner point of $\pi(\cdot)$ is called the optimal corner point.*

In the two-user case, for example, if the optimal solution is achieved at the red subregion between A and R_2 axis, then the optimal decoding order is $1 \to 2$ and A is the optimal corner point. It can be seen that an optimal decoding order may not exist in the dual MIMO-MAC. For example, For example, if the optimal solution is achieved at the line segment AB in Fig. 1(b), then none of the two decoding orders is optimal. However, if there exists an optimal decoding order, the following theorem shows the uniqueness of this optimal decoding order.

Theorem 1 (Uniqueness of Optimal Decoding Order)**.** *If there exists an optimal decoding order in a dual MIMO-MAC, it must be unique.*

Proof. It is obvious that the objective of (3) is strictly concave and the feasible region is convex with respect to $\mathbf{R}$ and non-empty. Thus, if an optimal decoding order exists, it must be unique. $\square$

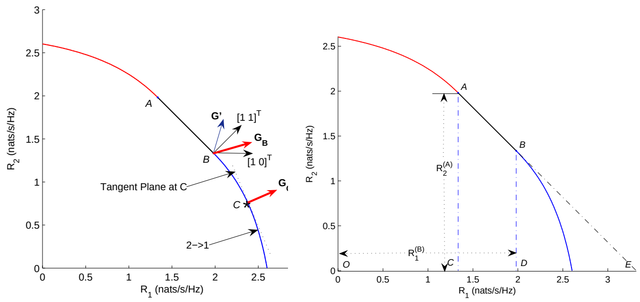
IV. Corner Point Optimality Conditions

Noting that different parts on the boundary of a dual MIMO-MAC capacity region may correspond to different decoding orders, we need to consider all possible decoding orders when optimizing a certain objective function. Since there are $K!$ possible decoding orders for a K-user dual MIMO-MAC, finding the optimal decoding order would quickly become intractable as the number of users gets large. In order to avoid blindly enumerating all possible decoding orders, we first study what conditions the optimal decoding order must satisfy. To reveal geometrical insights, we will first consider a two-user dual MIMO-MAC as an illustrative example and subsequently generalize the results to the case of K users.

A. Two-User Case

Since the log function is monotonically increasing, we must have that the optimal WPF solution is achieved on the boundary. Fig. 3(a) shows the boundary of a two-user dual MIMO-MAC.

Since all weights and data rates are positive, the gradient of every point in the capacity region, which is in the form of $[\frac{w_1}{R_1}\ \frac{w_2}{R_2}\ \ldots,\ \frac{w_K}{R_K}]$, must lie in the positive orthant. Suppose that the optimal WPF is achieved at point C where user 2 is decoded first. Then, the gradient of the objective function at corner point B, denoted as $\mathbf{G}_B$, must be contained in the *cone* formed by vectors $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ as shown in Fig. 3(a). Otherwise, if the gradient is outside of



(a) Gradients and the cone. (b) (7) and (8) not both satisfied.

Fig. 3. An example of a two-user dual MIMO-MAC capacity region illustrating the corner point optimality conditions.

the cone, as for $\mathbf{G}'$ in Fig. 3(a), then $\mathbf{G}'$, which points to the increasing objective contours, would have led the search of the optimal WPF to move away from the subregion where user 2 is decoded first, a contradiction. Let $R_1^{(B)}$ and $R_2^{(B)}$ denote the rates at corner point B. The above geometrical fact can be stated as the following linear combination:

$$\mathbf{G}_B = \begin{bmatrix} \frac{w_1}{R_1^{(B)}} \\ \frac{w_2}{R_2^{(B)}} \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{5}$$

where $\alpha_1, \alpha_2 \geq 0$. Solving (5), we have

$$\begin{cases} \alpha_1 = \frac{w_1}{R_1^{(B)}} - \frac{w_2}{R_2^{(B)}} \\ \alpha_2 = \frac{w_2}{R_2^{(B)}}. \end{cases} \tag{6}$$

Note that in (6), α_2 is non-negative. Thus, in order for (6) to be feasible, we must have $\alpha_1 \geq 0$, i.e., $\frac{w_1}{R_1^{(B)}} - \frac{w_2}{R_2^{(B)}} \geq 0$. This can be further rewritten as

$$\frac{R_1^{(B)}}{R_2^{(B)}} \leq \frac{w_1}{w_2}. \tag{7}$$

Similarly, for optimality at corner point A (optimal solution achieved at some other point C' corresponding to decoding user 1 first), we have

$$\frac{R_2^{(A)}}{R_1^{(A)}} \leq \frac{w_2}{w_1}. \tag{8}$$

Conditions (7) and (8) correspond to decoding orders $2 \rightarrow 1$ and $1 \rightarrow 2$ being optimal, respectively. However, there are still two open questions:

1) Is it possible that none of (7) and (8) is satisfied?
2) Is it possible that both (7) and (8) are satisfied?

The answer to the first question is yes. When (7) and (8) are violated simultaneously, the optimal solution must be located somewhere on the line segment AB. In this case, the gradient direction at the optimal solution must be perpendicular to line segment AB, i.e. $[\frac{w_1}{R_1^*}\ \frac{w_2}{R_2^*}]^T = \alpha[1\ 1]^T = 0$. With $R_1^* + R_2^* = R_{\text{Sum}}$, where R_{Sum} denotes the maximum achievable sum rate, we can derive the closed-form solution of R_1^* and R_2^* as follows:

$$\begin{cases} R_1^* = \frac{w_1}{w_1+w_2} R_{\text{Sum}} \\ R_2^* = \frac{w_2}{w_1+w_2} R_{\text{Sum}}. \end{cases}$$

The answer to the second question is no. To show this, we can extend the line segment AB to cross the R_1 axis at point E, as shown in Fig. 3(b). Also, draw two line segments AC and BD, which are perpendicular to the R_1 axis. It is obvious that the lengths $|AC| = R_2^{(A)}$ and $|OD| = R_1^{(B)}$. Now, by contradiction, suppose that it is possible to simultaneously satisfy (7) and (8). Since (7) and (8) can be rewritten as

$$R_1^{(B)} \leq \frac{w_1}{w_1+w_2} R_{\text{Sum}}, \quad \text{and} \quad R_2^{(A)} \leq \frac{w_2}{w_1+w_2} R_{\text{Sum}},$$

it follows that

$$R_1^{(B)} + R_2^{(A)} \leq R_{\text{Sum}}. \tag{9}$$

From Fig. 3(b), we see that $R_{\text{Sum}} = |OD| + |DE| = R_1^{(B)} + |DE|$. Also, note that $|DE| = |BD|$ and $|BD| < |AC|$. It follows that $R_{\text{Sum}} < R_1^{(B)} + |AC| = R_1^{(B)} + R_2^{(A)}$, a contradiction to (9).

B. *K-User Case*

The corner point optimality conditions for a general K-user case can be derived by extending the two-user case. We state the corner point optimality conditions in Theorem 2.

Theorem 2 (Corner Point Optimality Conditions)**.** *In a K-user dual MIMO-MAC channel, a decoding order $\pi^*(i) \in \{1, 2, \ldots, K\}$, $i = 1, 2, \ldots, K$, is the optimal decoding order if and only if $\pi^*(\cdot)$ and the data rates at its corresponding corner point, denoted by O, satisfies*

$$\frac{R_{\pi^*(i+1)}^{(O)}}{R_{\pi^*(i)}^{(O)}} \leq \frac{w_{\pi^*(i+1)}}{w_{\pi^*(i)}}, \quad \textit{for } i = 1, \ldots, K-1. \tag{10}$$

Proof. First, it is not difficult to observe that, if the optimal corner point exists, the optimal corner point must have the largest objective value among all corner points. Otherwise, there would have been an improving direction from the optimal corner point to another corner point. It is evident that this improving direction points away from the optimal subregion, which is contradict to the fact that the optimal solution is achieved in the optimal subregion. As a result, we can simply consider the beveled box that corresponds to the $K!$ corner points.

Suppose that the optimal corner point is O and the optimal decoding order is $\pi^*(\cdot)$. Since Problem (3) is convex with Slater constraint qualification holds, we have that KKT condition is both necessary and sufficient. The active rate constraints at O are

$$\sum_{i=j}^{K} R_{\pi^*(i)}^{(O)} \leq \log \left| \mathbf{I} + \sum_{i=j}^{K} \mathbf{H}_{\pi^*(i)}^{\dagger} \mathbf{Q}_{\pi^*(i)}^{*} \mathbf{H}_{\pi^*(i)} \right|,$$

$j = 1, \ldots, K$, where $\mathbf{Q}_i^*, i = 1, \ldots, K$, are the optimal input covariance matrices that achieves the maximum sum rate.

Then, by KKT condition, we must have that

$$\begin{bmatrix} \frac{w_{\pi^*(1)}}{R^{(O)}_{\pi^*(1)}} \\ \vdots \\ \frac{w_{\pi^*(K-1)}}{R^{(O)}_{\pi^*(K-1)}} \\ \frac{w_{\pi^*(K)}}{R^{(O)}_{\pi^*(K)}} \end{bmatrix} = u_1 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \cdots + u_K \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}, \tag{11}$$

where $u_i \geq 0, \forall i$. Solving for u_i in (11), we have

$$u_K = \frac{w_{\pi^*(1)}}{R^{(O)}_{\pi^*(1)}} \quad \text{and} \quad u_{K-i} = \frac{w_{\pi^*(i+1)}}{R^{(O)}_{\pi^*(i+1)}} - \frac{w_{\pi^*(i)}}{R^{(O)}_{\pi^*(i)}},$$

$i = 1, \ldots, K-1$. Since $u_i \geq 0$, it then follows that

$$\frac{R^{(O)}_{\pi^*(i+1)}}{R^{(O)}_{\pi^*(i)}} \leq \frac{w_{\pi^*(i+1)}}{w_{\pi^*(i)}}, \quad \text{for } i = 1, \ldots, K-1.$$

□

By setting all weights in (10) to 1, we have the following result.

Corollary 1. *In a K-user dual MIMO-MAC channel, a decoding order $\pi(i) \in \{1, 2, \ldots, K\}$, $i = 1, 2, \ldots, K$, is optimal for the maximum log sum rate problem if and only if $\pi(\cdot)$ and the data rates at its corresponding corner point, denoted by A, satisfies*

$$R^{(A)}_{\pi(1)} \geq R^{(A)}_{\pi(2)} \geq \ldots \geq R^{(A)}_{\pi(K)}. \tag{12}$$

On the other hand, if none of the corner points satisfies (10), then there is no such decoding order along which we can achieve the maximum WPF in MIMO-BC. In this case, we have the following theorem.

Theorem 3. *If the optimal decoding order does not exist for a K-user MIMO-BC, then the optimal WPF solution must be located on one of the $\sum_{i=2}^{K} \binom{K}{i}(K-i+1)!$ time-sharing subregions on the boundary. Without loss of generality, suppose that the optimal WPF solution is achieved at a time-sharing subregion that involves users $1, 2, \ldots, i$. Then we must have*

$$R_j = \frac{w_j}{\sum_{k=1}^{i} w_k} R_{\text{Sum}}, \quad j = 1, 2, \ldots, i, \tag{13}$$

where $R_{\text{Sum}} = \sum_{j=1}^{i} R_j$.

Proof. The first part of the theorem follows immediately from Lemma 1. To show that second part, we note that the gradient direction at the optimal point must be orthogonal to the hyperplane $\sum_{k=1}^{i} R_k \leq \log \left| \mathbf{I} + \sum_{k=1}^{i} \mathbf{H}_k^{\dagger} \mathbf{Q}_k \mathbf{H}_k \right|$. This means that

$$\begin{bmatrix} \frac{w_1}{R_1^*} & \cdots & \frac{w_i}{R_i^*} \end{bmatrix}^T = \alpha \mathbf{1}, \tag{14}$$

for some $\alpha > 0$. Thus, we have

$$\frac{w_1}{R_1^*} = \frac{w_2}{R_2^*} = \cdots = \frac{w_i}{R_i^*}.$$

It then follows that

$$R_j^* = \frac{w_j}{\sum_{k=1}^{i} w_k} R_{\text{Sum}}.$$

□

In order to determine the optimal decoding order, we need to further know the data rates for all users at each corner point, which can be computed as follows. First, solve the maximum sum rate problem to get the optimal set of input covariance matrices $\mathbf{Q}_1^*, \mathbf{Q}_2^*, \ldots, \mathbf{Q}_K^*$ (see [7] for details). Then, for a corner point A with decoding order $\pi(\cdot)$, the data rates can be computed as follows:

$$R^{(A)}_{\pi(K)} = \log \left| \mathbf{I} + \mathbf{H}^{\dagger}_{\pi(K)} \mathbf{Q}^*_{\pi(K)} \mathbf{H}_{\pi(K)} \right| \tag{15}$$

and

$$\begin{aligned} R^{(A)}_{\pi(i)} = {} & \log \left| \mathbf{I} + \textstyle\sum_{j=i}^{K} \mathbf{H}^{\dagger}_{\pi(j)} \mathbf{Q}^*_{\pi(j)} \mathbf{H}_{\pi(j)} \right| \\ & - \log \left| \mathbf{I} + \textstyle\sum_{j=i+1}^{K} \mathbf{H}^{\dagger}_{\pi(j)} \mathbf{Q}^*_{\pi(j)} \mathbf{H}_{\pi(j)} \right|, \end{aligned} \tag{16}$$

for $i = 1, 2, \ldots, K-1$.

V. Determining Optimal Decoding Order

In fact, a brute force search based on Theorem 2 can be used to determine the optimal decoding order. However, since there are $K!$ corner points, such a brute force search is arduous for large-sized networks. In this section, we exploit the special geometric structure of the capacity region and take advantage of the gradient information at each corner point to design an efficient algorithm called "iterative gradient sorting" (IGS).

A. Iterative Gradient Sorting

The basic idea of IGS is to proactively look for the optimal corner point rather than blindly enumerate all of them. In IGS, we start from an arbitrarily selected corner point and use the gradient at this corner point as an *approximation* of the true gradient at the optimal corner point. It is not difficult to see that if we sort the entries of this approximate gradient, we would have an approximate decoding order of the true optimal decoding order. Thus, if the approximate gradient direction is close enough to the true gradient direction, we can expect that the resultant approximate decoding order is close to the optimal decoding order. To reveal geometrical insights, let us first consider a three-user example as shown in Fig. 4.

Fig. 4 shows the beveled box for which the hexagon is coincident with the hexagon of the capacity region, i.e., the six corner points of the hexagon are exactly the same corner points of the capacity region. Suppose that the optimal decoding order is achieved at corner point O with $\pi^* = 1 \rightarrow 2 \rightarrow 3$. Suppose, also, that we arbitrarily start at some corner point, say A as shown in Fig. 4. We use the gradient at A, denoted by $\mathbf{G}_A$, as an approximation of $\mathbf{G}_O$. Now, we sort the entries of $\mathbf{G}_A$ and denote the resultant ordering as π'. If $\mathbf{G}_A$ is close to $\mathbf{G}_O$, we should have that π' is also close to π^*. We can use the corner point optimality conditions to check if π' is indeed the optimal decoding order. If yes, then we are done. Otherwise, this π' gives us a new non-optimal corner point, say B as

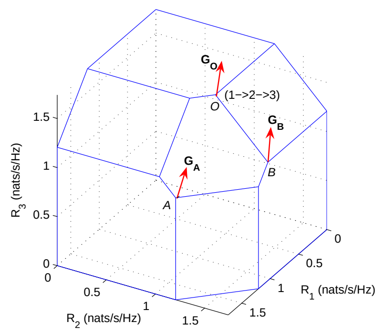


Fig. 4. Example of iterative gradient sorting.

shown in Fig. 4. Again, we sort the entries of the gradient $\mathbf{G}_B$ at B. This sorting process continues iteratively until we find the optimal decoding order. The IGS process is summarized in Algorithm 1. It is worth pointing out that IGS can be

Algorithm 1 Iterative Gradient Sorting (IGS)

1. Compute the set of input covariance matrices $\mathbf{Q}_1^*, \ldots, \mathbf{Q}_K^*$ that maximizes the sum rate capacity of the dual MIMO-MAC channel. Arbitrarily pick a decoding order to start.
2. For the current decoding order $\pi(\cdot)$, use (10) to test whether or not it is optimal. If yes, return $\pi(\cdot)$ and stop.
3. Compute the gradient at the corner point corresponding to $\pi(\cdot)$, denoted by $\mathbf{G}$. Sort the entries of $\mathbf{G}$ to have a new ordering π'. Let $\pi = \pi'$ and repeat Step 2.

geometrically interpreted as "sequential gradient projections." Consider the projection of $\mathbf{G}_A$ onto the $K-1$ dimensional hyperplane $\sum_{i=1}^{K} R_{\pi(i)} \leq \log_2 |\mathbf{I}+\mathbf{H}_{\pi(i)}^{\dagger}\mathbf{Q}_{\pi(i)}^{*}\mathbf{H}_{\pi(i)}|$. Denote the projected gradient as $\mathbf{G}_A^{(K)}$ and it can be readily verified that $\mathbf{G}_A^{(K)} = \mathbf{G}_A - \frac{\langle \mathbf{G}_A, \mathbf{1} \rangle}{K}\mathbf{1}$. IGS can be geometrically interpreted as follows: Start from A and move along $\mathbf{G}_A^{(K)}$ until blocked by one user's rate constraint. Then, compute the gradient projection onto the $K-2$ dimensional hyperplane and continue until the hyperplane becomes one dimensional. Due to space limitation, we refer readers to [11] for more details.

B. Convergence of IGS and Complexity Analysis

We now show that if the geometric structure of the dual MIMO-MAC capacity region satisfies certain mild conditions, IGS is guaranteed to find the optimal corner point and converges in polynomial time. The convergence of IGS hinges upon the following theorem.

Theorem 4. *Start from a corner point, denoted by A, and sort the entries of its gradient, denoted by $\mathbf{G}_A$. Denote the resultant ordering as π'. We have $\pi'(1) = \pi^*(1)$, where $\pi^*(\cdot)$ denotes the optimal decoding order, if*

$$\frac{w_{\pi^*(1)}}{R_{\pi^*(1)}^{(A)}} \leq \frac{w_{\pi^*(j)}}{\bar{R}_{\pi^*(j)}}, \quad j = 1, \ldots, K, j \neq \pi^*(1), \tag{17}$$

where $\bar{R}_{\pi^*(j)} = \log \left| \mathbf{I} + \mathbf{H}_{\pi^*(j)}^{\dagger}\mathbf{Q}_{\pi^*(j)}\mathbf{H}_{\pi^*(j)} \right|$ *is the upper bound of* $R_{\pi^*(j)}$.

Proof. Suppose that the optimal corner point is O and the corresponding decoding order is $\pi^*(\cdot)$. At point O, we must have that $R_{\pi^*(1)}^{(O)}$ is at the lower bound of $R_{\pi^*(1)}$. This is because from (16), we can have that

$$R_{\pi(i)}^{(A)} = \log \left| \mathbf{I} + \hat{\mathbf{H}}_{\pi(i)}^{\dagger}\mathbf{Q}_{\pi(i)}^{*}\hat{\mathbf{H}}_{\pi(i)} \right|,$$

where $\hat{\mathbf{H}}_{\pi(j)}$ is the effective channel gain matrix that represents the interference plus noise experienced at user $\pi(j)$ and is computed as

$$\hat{\mathbf{H}}_{\pi(i)} = \left(\mathbf{I} + \sum_{j=i+1}^{K} \mathbf{H}_{\pi(j)}^{\dagger}\mathbf{Q}_{\pi(j)}^{*}\mathbf{H}_{\pi(j)} \right)^{-\frac{1}{2}} \mathbf{H}_{\pi(i)}. \tag{18}$$

Since $R_{\pi^*(1)}$ is the first one to be decoded at point O, from (18), we see that $R_{\pi^*(1)}^{(O)}$ has the largest number of interferers. Thus, $R_{\pi^*(1)}^{(O)}$ is at the lower bound of $R_{\pi^*(1)}$. This also implies that if we move away from O to another corner point, say A, $R_{\pi^*(1)}$ is non-decreasing. It then follows that

$$\frac{w_{\pi^*(1)}}{R_{\pi^*(1)}^{(A)}} \leq \frac{w_{\pi^*(1)}}{R_{\pi^*(1)}^{(O)}}.$$

To ensure that $\frac{w_{\pi^*(1)}}{R_{\pi^*(1)}^{(A)}}$ remains the smallest entry in the gradient at A, we must have that

$$\frac{w_{\pi^*(1)}}{R_{\pi^*(1)}^{(A)}} \leq \frac{w_{\pi^*(j)}}{\bar{R}_{\pi^*(j)}}, \quad j = 1, \ldots, K, j \neq \pi^*(1).$$

That is, even if $R_{\pi^*(j)}$ is at its upper bound, its gradient entry remain larger than that of $R_{\pi^*(1)}$. □

Remark 1. It is worth pointing out that the conditions in Theorem 4 is not very restrictive. This is because when moving from O to A, the changes of rates scale in logarithmic order, which does not result in dramatic changes. Therefore, in most cases, the conditions in Theorem 4 can be easily satisfied. Also, this implies that in general, the gradients of non-optimal corner points are usually good approximations of the true optimal gradient of the optimal corner point.

Along the same line of the proof of Theorem 4, we have the following corollary.

Corollary 2. *During the k^{th} round of gradient projection for a dual MIMO-MAC with $K-k+1$ users, We find $\pi'(1)$ correctly if*

$$\frac{w_{\pi^*(1)}}{R_{\pi^*(1)}^{(A)}} \leq \frac{w_{\pi^*(j)}}{\bar{R}_{\pi^*(j)}}, \quad j = 1, \ldots, K-k+1, j \neq \pi^*(1), \tag{19}$$

From Theorem 4 and Corollary 2, we see that we can determine at least one position correctly if the capacity region satisfies the conditions. Since there are K positions in total, IGS is guaranteed to terminate in finite number of times.

It can be seen from Theorem 4 and Corollary 2 that for a K-user dual MIMO-MAC with optimal decoding order existing, we can determine one decoding position correctly after one round of gradient sorting. Thus, we need to perform at most $K-1$ rounds of sorting to determine the optimal decoding order. As a result, the complexity of IGS is $O((K-1)K \log K)$.

C. Numerical Results

We use a 15-user MIMO-BC example to demonstrate the efficacy of IGS. It is easy to verify that there are $15! \approx 1.3077 \times 10^{12}$ corner points in total, which means it is not viable to use a brute force search. As shown in Fig. 5, the nodes are indexed from 1 to 15 and are randomly distributed in a square region. The transmitter is located at the center. The transmitter and all the receivers are equipped with four antennas. For user 1 to user 15, the weights are 1, 1, 1.2, 0.7, 0.33, 0.25, 0.35, 0.2, 0.9, 5, 0.65, 0.8, 0.4, 0.78, and 1, respectively. In this example, the optimal decoding order in the dual MIMO-MAC exists and is: $4 \rightarrow 12 \rightarrow 9 \rightarrow 15 \rightarrow 11 \rightarrow 3 \rightarrow 7 \rightarrow 5 \rightarrow 6 \rightarrow 8 \rightarrow 13 \rightarrow 14 \rightarrow 1 \rightarrow 10 \rightarrow 2$.

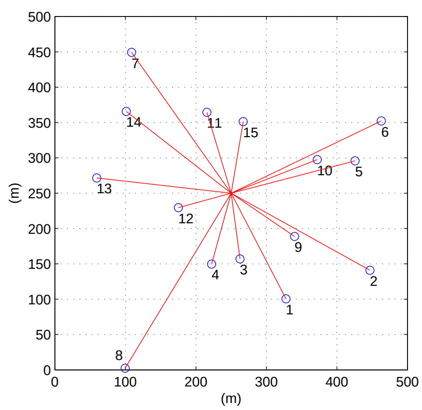


Fig. 5. Network topology of a 15-user MIMO-BC.

To test IGS's efficiency and robustness, we start from two completely reversed initial orders $1, 2, \ldots, 15$ and $15, 14, \ldots, 1$, such that they are far away from each other. For convenience, we call them Case 1 and Case 2, respectively. The result of IGS in each iteration is shown in Table I. For both cases, IGS found the optimal decoding order in just two iterations. The running time is less than 1 second for both cases. We notice that after the first iteration, the first 5 and 4 positions are already correct in Case 1 and Case 2, respectively.

TABLE I
IGS FOR A 15-USER MIMO-BC EXAMPLE

	Case 1			Case 2		
No.	Initial	Iter. 1	Iter. 2	Initial	Iter. 1	Iter. 2
1	1	4	4	15	4	4
2	2	12	12	14	12	12
3	3	9	9	13	9	9
4	4	15	15	12	15	15
5	5	11	11	11	3	11
6	6	13	3	10	7	3
7	7	14	7	9	5	7
8	8	6	5	8	6	5
9	9	5	6	7	1	6
10	10	7	8	6	8	8
11	11	10	13	5	11	13
12	12	8	14	4	2	14
13	13	3	1	3	14	1
14	14	1	10	2	10	10
15	15	2	2	1	13	2

VI. Optimal Input Covariance Matrices

After determining the optimal decoding order $\pi^*(\cdot)$, the next step is to compute the optimal covariance matrices. However, from (15) and (16), we see that even with the knowledge of the optimal decoding order, (3) is still a nonconvex optimization problem, which is difficult to solve. In this paper, we propose an efficient algorithm based on conjugate gradient projection (CGP) to determine a local optimal solution. CGP utilizes the important concept of Hessian-conjugate direction to deflect the gradient direction appropriately so as to achieve an asymptotic K-step superlinear convergence rate [12], similar to that of the quasi-Newton methods (e.g., BFGS method). The framework of CGP is shown in Algorithm 2. We adopt the Armijo Rule in-

Algorithm 2 Conjugate Gradient Projection Method

Initialization:
Choose $\mathbf{Q}^{(0)}_{\pi^*} = [\mathbf{Q}^{(0)}_{\pi^*(1)}, \mathbf{Q}^{(0)}_{\pi^*(2)}, \ldots, \mathbf{Q}^{(0)}_{\pi^*(K)}]^T$. Let $k = 0$.
Main Loop:
1. Calculate the conjugate gradients $\mathbf{G}^{(k)}_{\pi(i)}$, $i = 1, 2, \ldots, K$.
2. Choose an appropriate step size s_k. Let $\mathbf{Q}'^{(k)}_{\pi^*(i)} = \mathbf{Q}^{(k)}_{\pi^*(i)} + s_k \mathbf{G}^{(k)}_{\pi^*(i)}$, for $i = 1, 2, \ldots, K$.
3. Let $\bar{\mathbf{Q}}^{(k)}_{\pi^*}$ be the projection of $\mathbf{Q}'^{(k)}_{\pi^*}$ onto $\Omega_+(P)$.
4. Choose an appropriate step size α_k. Let $\mathbf{Q}^{(k+1)}_{\pi^*(i)} = \mathbf{Q}^{(k)}_{\pi^*(i)} + \alpha_k (\bar{\mathbf{Q}}^{(k)}_{\pi^*(i)} - \mathbf{Q}^{(k)}_{\pi^*(i)})$, $i = 1, 2, \ldots, K$.
5. $k = k+1$. If the maximum absolute value of the elements in $\mathbf{Q}^{(k)}_{\pi^*(i)} - \mathbf{Q}^{(k-1)}_{\pi^*(i)} < \epsilon$, for $i = 1, 2, \ldots, L$, then stop; else go to step 1.

exact line search method to avoid excessive objective function evaluations [12]. For convenience, we use $F(\mathbf{Q})$ to represent the objective function in (3), where $\mathbf{Q} = (\mathbf{Q}_1, \ldots, \mathbf{Q}_K)$ denotes the set of covariance matrices at a node.

1) Computing the Conjugate Gradients: From (15) and (16), $F(\mathbf{Q}_{\pi^*})$ can be written as

$$F(\mathbf{Q}_{\pi^*}) = w_{\pi^*(K)} \log \left[\log \left| \mathbf{I} + \mathbf{H}^{\dagger}_{\pi^*(K)} \mathbf{Q}_{\pi^*(K)} \mathbf{H}_{\pi^*(K)} \right| \right] + \sum_{i=1}^{K-1} w_{\pi^*(i)} \log \left[\log \left| \mathbf{I} + \sum_{j=i}^{K} \mathbf{H}^{\dagger}_{\pi^*(j)} \mathbf{Q}_{\pi^*(j)} \mathbf{H}_{\pi^*(j)} \right| - \log \left| \mathbf{I} + \sum_{j=i+1}^{K} \mathbf{H}^{\dagger}_{\pi^*(j)} \mathbf{Q}_{\pi^*(j)} \mathbf{H}_{\pi^*(j)} \right| \right]. \quad (20)$$

The gradient $\bar{\mathbf{G}}_{\pi^*(j)} \triangleq \nabla_{\mathbf{Q}_{\pi^*(j)}} F(\mathbf{Q}_{\pi^*})$ depends on the partial derivatives of $F(\mathbf{Q}_{\pi^*})$ with respect to $\mathbf{Q}_{\pi^*(j)}$. For convenience, we let $\mathbf{M}_i = \mathbf{I} + \sum_{j=i}^{K} \mathbf{H}^{\dagger}_{\pi^*(j)} \mathbf{Q}_{\pi^*(j)} \mathbf{H}_{\pi^*(j)}$. By using the formula $\frac{\partial \ln |\mathbf{A}+\mathbf{BXC}|}{\partial \mathbf{X}} = \left[\mathbf{C}(\mathbf{A}+\mathbf{BXC})^{-1}\mathbf{B} \right]^T$ [13], [14], we can compute the partial derivative of the i^{th} term in the summation of $F(\mathbf{Q}_{\pi^*})$ with respect to $\mathbf{Q}_{\pi^*(j)}$, $j \geq i$, as follows:

$$\frac{\partial F^{(i)}}{\partial \mathbf{Q}_{\pi^*(j)}} \triangleq \frac{\partial}{\partial \mathbf{Q}_{\pi^*(j)}} w_{\pi^*(i)} \log \left(\log |\mathbf{M}_i| - \log |\mathbf{M}_{i+1}| \right) = w_{\pi^*(i)} \frac{\left(\mathbf{H}_{\pi^*(j)} \left[\mathbf{M}_i^{-1} - \mathbf{M}_{i+1}^{-1} \right] \mathbf{H}^{\dagger}_{\pi^*(j)} \right)^T}{\log |\mathbf{M}_i| - \log |\mathbf{M}_{i+1}|}.$$

To compute the gradient of $F(\mathbf{Q}_{\pi^*})$ with respect to $\mathbf{Q}_{\pi^*(j)}$, we note that only the first j terms in $F(\mathbf{Q}_{\pi^*})$ involve $\mathbf{Q}_{\pi^*(j)}$. From the definition $\nabla_z f(z) = 2(\partial f(z)/\partial z)^*$ [15], we have, for $1 \le j \le K$,

$$\bar{\mathbf{G}}_{\pi^*(j)} = \sum_{i=1}^{j} 2\left(\frac{\partial F^{(i)}}{\partial \mathbf{Q}_{\pi^*(j)}}\right)^* = w_{\pi^*(j)} \frac{\mathbf{H}_{\pi(j)}\mathbf{M}_j^{-1}\mathbf{H}_{\pi(j)}^\dagger}{\log|\mathbf{M}_j| - \log|\mathbf{M}_{j+1}|} + \sum_{i=1}^{j-1} w_{\pi^*(i)} \frac{\mathbf{H}_{\pi^*(j)}\left(\mathbf{M}_i^{-1} - \mathbf{M}_{i+1}^{-1}\right)\mathbf{H}_{\pi^*(j)}^\dagger}{\log|\mathbf{M}_i| - \log|\mathbf{M}_{i+1}|}. \tag{21}$$

It is important to point out that we can exploit the special structure in (21) to reduce the complexity in computing the gradients. Note that the most difficult part in computing $\bar{\mathbf{G}}_{\pi^*(j)}$ is the summation of the terms in $\mathbf{M}_j$. However, note that when j varies, most of the terms in the summation are still the same. Thus, we can maintain a running sum for $\mathbf{M}_i$, start out from $j = K$, and reduce j by one sequentially. As a result, only one new term is added to the running sum in each iteration, resulting in only one addition in each iteration.

The conjugate gradient direction in the m^{th} iteration can be computed as $\mathbf{G}_{\pi^*(j)}^{(m)} = \bar{\mathbf{G}}_{\pi^*(i)}^{(m)} + \kappa_m \mathbf{G}_{\pi^*(i)}^{(m-1)}$. We adopt the Fletcher and Reeves' choice of deflection [12], which can be computed as

$$\kappa_m = \frac{\|\bar{\mathbf{G}}_{\pi^*(j)}^{(m)}\|^2}{\|\bar{\mathbf{G}}_{\pi^*(j)}^{(m-1)}\|^2}. \tag{22}$$

The purpose of deflecting the gradient using (22) is to find the Hessian-conjugate direction that tend to reduce the "zigzagging" phenomenon encountered in the conventional gradient projection method; achieve an asymptotic K-step superlinear convergence rate under certain regulation conditions [12]; and without actually storing a large Hessian approximation matrix as in quasi-Newton methods.

2) Projection onto $\Omega_+(P)$: Noting from (21) that $\mathbf{G}_{\pi^*(j)}$ is Hermitian, we have that $\mathbf{Q}_{\pi^*(j)}^{'(k)} = \mathbf{Q}_{\pi^*(j)}^{(k)} + s_k \mathbf{G}_{\pi^*(j)}^{(k)}$ is Hermitian as well. Then, the projection problem becomes how to simultaneously project K Hermitian matrices onto the set

$$\Omega_+(P) \triangleq \left\{ \mathbf{Q}_i \,\middle|\, \begin{array}{l} \sum_i \mathrm{Tr}\{\mathbf{Q}_i\} \le P, \\ \mathbf{Q}_i \succeq 0, \; i = 1, 2, \ldots, K \end{array} \right\}.$$

This problem belongs to the class of "matrix nearness problems" [16], [17], which is not easy to solve in general. However, by exploiting the special structure, we are able to design a polynomial-time algorithm. We construct a block diagonal matrix $\mathbf{D} = \mathrm{Diag}\{\mathbf{Q}_{\pi^*(1)} \ldots \mathbf{Q}_{\pi^*(K)}\} \in \mathbb{C}^{(K\cdot n_r)\times(K\cdot n_r)}$. It is easy to recognize that $\mathbf{Q}_{\pi^*(j)} \in \Omega_+(P)$, $j = 1, \ldots, K$, if $\mathrm{Tr}(\mathbf{D}) = \sum_{j=1}^K \mathrm{Tr}\left(\mathbf{Q}_{\pi^*(j)}\right) \le P$ and $\mathbf{D} \succeq 0$. We use Frobenius norm, denoted by $\|\cdot\|_F$, as the matrix distance metric. Thus, given a block diagonal matrix $\mathbf{D}$, we wish to find a matrix $\tilde{\mathbf{D}} \in \Omega_+(P)$ such that $\tilde{\mathbf{D}}$ minimizes $\|\tilde{\mathbf{D}} - \mathbf{D}\|_F$. For more convenient algebraic manipulations, we instead study the following equivalent optimization problem:

$$\begin{array}{ll} \text{Minimize} & \frac{1}{2}\|\tilde{\mathbf{D}} - \mathbf{D}\|_F^2 \\ \text{subject to} & \mathrm{Tr}(\tilde{\mathbf{D}}) \le P, \; \tilde{\mathbf{D}} \succeq 0. \end{array} \tag{23}$$

It is readily verifiable that (23) is a convex minimization problem. So we can solve it through its Lagrangian dual problem (assuming a suitable constraint qualification [12]). Associating Hermitian matrix $\mathbf{\Pi}$ to the constraint $\tilde{\mathbf{D}} \succeq 0$ and μ to the constraint $\mathrm{Tr}(\tilde{\mathbf{D}}) \le P$, we can write the Lagrangian as $g(\mathbf{\Pi}, \mu) = \min_{\tilde{\mathbf{D}}}\{(1/2)\|\tilde{\mathbf{D}} - \mathbf{D}\|_F^2 - \mathrm{Tr}(\mathbf{\Pi}^\dagger \tilde{\mathbf{D}}) + \mu(\mathrm{Tr}(\tilde{\mathbf{D}}) - P)\}$. After some simplifications (see [11]), the Lagrangian dual problem can be written as

$$\begin{array}{ll} \text{Maximize} & -\frac{1}{2}\|\mathbf{D} - \mu\mathbf{I} + \mathbf{\Pi}\|_F^2 - \mu P + \frac{1}{2}\|\mathbf{D}\|_F^2 \\ \text{subject to} & \mathbf{\Pi} \succeq 0, \mu \ge 0. \end{array} \tag{24}$$

After solving (24), we can have the optimal solution to (23) as $\tilde{\mathbf{D}}^* = \mathbf{D} - \mu^*\mathbf{I} + \mathbf{\Pi}^*$, where μ^* and $\mathbf{\Pi}^*$ are the optimal dual solutions to Lagrangian dual problem in (24). From Moreau Decomposition [18], we immediately have $\min_{\mathbf{\Pi}} \|\mathbf{D} - \mu\mathbf{I} + \mathbf{\Pi}\|_F = (\mathbf{D} - \mu\mathbf{I})_+$, where the operation $(\mathbf{A})_+$ means performing eigenvalue decomposition on matrix $\mathbf{A}$, keeping the eigenvector matrix unchanged, setting all non-positive eigenvalues to zero, and then multiplying back. Thus, the matrix variable $\mathbf{\Pi}$ in the Lagrangian dual problem can be removed and the Lagrangian dual problem can be rewritten as

$$\begin{array}{ll} \text{Maximize} & \psi(\mu) \triangleq -\frac{1}{2}\left\|(\mathbf{D} - \mu\mathbf{I})_+\right\|_F^2 - \mu P \\ \text{subject to} & \mu \ge 0. \end{array} \tag{25}$$

Suppose that after performing eigenvalue decomposition on $\mathbf{D}$, we have $\mathbf{D} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger$, where $\mathbf{\Lambda}$ is the diagonal matrix formed by the eigenvalues of $\mathbf{D}$ and $\mathbf{U}$ is the unitary matrix formed by the corresponding eigenvectors. Since $\mathbf{U}$ is unitary, we have $(\mathbf{D} - \mu\mathbf{I})_+ = \mathbf{U}(\mathbf{\Lambda} - \mu\mathbf{I})_+ \mathbf{U}^\dagger$. It then follows that $\left\|(\mathbf{D} - \mu\mathbf{I})_+\right\|_F^2 = \left\|(\mathbf{\Lambda} - \mu\mathbf{I})_+\right\|_F^2$. Denote the eigenvalues in $\mathbf{\Lambda}$ by λ_i, $i = 1, \ldots, K \times n_r$, and suppose that we sort them in non-increasing order such that $\mathbf{\Lambda} = \mathrm{Diag}\{\lambda_1\ \lambda_2 \ldots \lambda_{K\cdot n_r}\}$, where $\lambda_1 \ge \ldots \ge \lambda_{K\cdot n_r}$. It then follows that $\left\|(\mathbf{\Lambda} - \mu\mathbf{I})_+\right\|_F^2 = \sum_{j=1}^{K\cdot n_r} \left(\max\{0, \lambda_j - \mu\}\right)^2$. So, we can rewrite $\psi(\mu)$ as $\psi(\mu) = -\frac{1}{2}\sum_{j=1}^{K\cdot n_r}\left(\max\{0, \lambda_j - \mu\}\right)^2 - \mu P$. From the special structure of $\psi(\mu)$, we are able to design a polynomial time algorithm. Due to space limitation, we refer readers to [11] for more details. The projection of $\mathbf{D}$ onto $\Omega_+(P)$ is summarized in Algorithm 3.

Algorithm 3 Projection onto $\Omega_+(P)$

Initiation:

1. Construct a block diagonal matrix $\mathbf{D}$. Perform eigenvalue decomposition $\mathbf{D} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger$, sort the eigenvalues in non-increasing order.
2. Introduce $\lambda_0 = \infty$ and $\lambda_{K\cdot n_t+1} = -\infty$. Let $\hat{I} = 0$. Let the endpoint objective value $\psi_{\hat{I}}(\lambda_0) = 0$, $\phi^* = \psi_{\hat{I}}(\lambda_0)$, and $\mu^* = \lambda_0$.

Main Loop:

1. If $\hat{I} > K\cdot n_r$, go to the final step; else let $\mu_{\hat{I}}^* = (\sum_{j=1}^{\hat{I}} \lambda_j - P)/\hat{I}$.
2. If $\mu_{\hat{I}}^* \in [\lambda_{\hat{I}+1}, \lambda_{\hat{I}}] \cap \mathbb{R}_+$, then let $\mu^* = \mu_{\hat{I}}^*$ and go to the final step.
3. Compute $\psi_{\hat{I}}(\lambda_{\hat{I}+1})$. If $\psi_{\hat{I}}(\lambda_{\hat{I}+1}) < \phi^*$, then go to the final step; else let $\mu^* = \lambda_{\hat{I}+1}$, $\phi^* = \psi_{\hat{I}}(\lambda_{\hat{I}+1})$, $\hat{I} = \hat{I} + 1$ and continue.

Final Step: Compute $\tilde{\mathbf{D}}$ as $\tilde{\mathbf{D}} = \mathbf{U}(\mathbf{\Lambda} - \mu^*\mathbf{I})_+ \mathbf{U}^\dagger$.

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3) Numerical Example: For the 15-user MIMO-BC example in Fig. 5, the convergence process of the CGP algorithm is plotted in Fig. 6. We start from the optimal decoding corner point (that achieves the maximum sum rate), which corresponds to an unfair rate vector. It can be seen that CGP takes only 35 iterations to converge. The data rates of the 15 users are plotted in Fig. 7, from which we can see that the data rates converge to a proportional fair status.

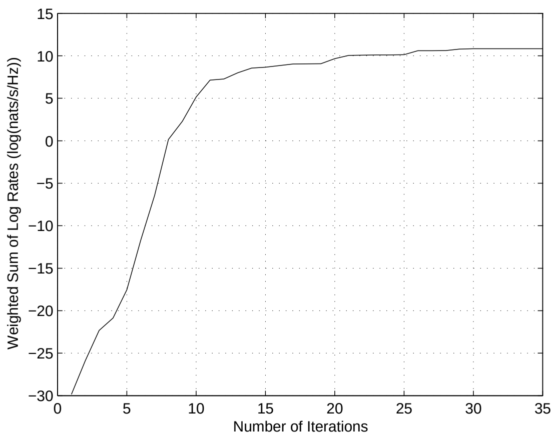

Fig. 6. Convergence behavior of the CGP algorithm.

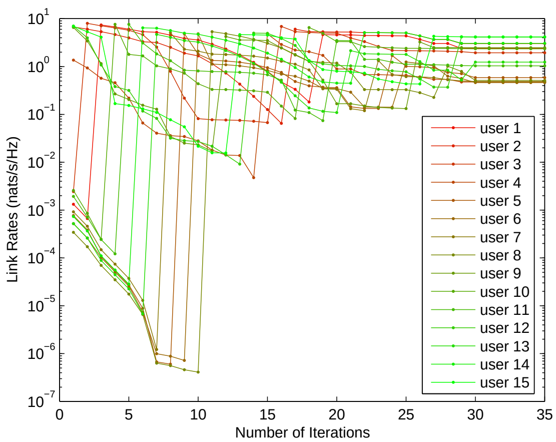

Fig. 7. Data rates converge to proportional fair status.

VII. Conclusion

In this paper, we studied how to determine the WPF capacity for MIMO-BC. Our main contributions are three-fold. First, we derived a set of optimality conditions that the optimal decoding order must satisfy. Second, based on the optimality conditions, we designed an efficient algorithm called iterative gradient sorting (IGS) to determine the optimal decoding order by iteratively sorting the gradient entries. We further showed that this method can be geometrically interpreted as sequential gradient projections. Third, we proposed an efficient algorithm based on conjugate gradient projection (CGP) for computing input covariance matrices to achieve the WPF capacity. Collectively, these results fill an important gap in dealing with fairness issues in MIMO-BC. In our future work, we will further study how to handle the nonconvex difficulty in computing the optimal input covariance matrices.

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